Exercice 2 *Disturbed harmonic oscillator - explanation of Eq.(18)*

Up to the first order in perturbation theory the n-th energy level wave function is given by

$$|\Psi_n(t)\rangle = |\Psi_n^{(0)}(t)\rangle + \sum_k a_{kn}^{(1)}(t)|\Psi_k^{(0)}(t)\rangle,$$

where

$$|\Psi_n^{(0)}(t)\rangle = |\Psi_n^{(0)}\rangle e^{\frac{i}{\hbar}E_n^{(0)}t}.$$

Substitution of this expression into Schroedinger equation gives

$$a_{kn}^{(1)}(t) = -\frac{i}{\hbar}\int_{-\infty}^{t}\langle\Psi_k^{(0)}(t)|\hat{V}(t)|\Psi_n^{(0)}(t)\rangle dt = -\frac{i}{\hbar}\int_{-\infty}^{t} V_{kn}(t)e^{i\omega_{kn}t}dt,$$

where $\omega_{kn} = (E_n^{(0)} - E_k^{(0)})/\hbar$. In the last expression one can integrate by parts, which leads to

$$a_{kn}^{(1)}(t) = -\left.\frac{V_{kn}(t)e^{i\omega_{kn}t}}{\hbar\omega_{kn}}\right|_{-\infty}^{t} + \int_{-\infty}^{t}\frac{\partial V_{kn}(t)}{\partial t}\frac{e^{i\omega_{kn}t}}{\hbar\omega_{kn}}dt.$$

Now, suppose the perturbation $\hat{V}(t)$ switches on at some moment $t = t_0$ and has an asymptotic value at large t, $\hat{V}(t) \to \hat{V}$, $t \to \infty$. For example, in our particular problem $\hat{V} = \texttt{const}$ for all $t > t_0$. Then, for the first order wave functions we have

$$|\Psi_n(t)\rangle = |\Psi_n^{(0)}(t)\rangle + \sum_k \frac{V_{kn}(t)e^{i\omega_{kn}t}}{\hbar\omega_{kn}}|\Psi_k^{(0)}(t)\rangle + \sum_k a_{kn}^{'(1)}(t)\Psi_k^{(0)}(t)\rangle,$$

where

$$a_{kn}^{'(1)}(t) = \int_{-\infty}^{t}\frac{\partial V_{kn}(t)}{\partial t}\frac{e^{i\omega_{kn}t}}{\hbar\omega_{kn}}dt,$$

or, writing explicitly the time-dependent parts of $|\Psi_n^{(0)}(t)\rangle$,

$$|\Psi_n(t)\rangle = |\Psi_n^{(0)}\rangle e^{\frac{i}{\hbar}E_n^{(0)}t} + \sum_k \frac{V_{kn}(t)}{E_n^{(0)} - E_k^{(0)}}|\Psi_k^{(0)}\rangle e^{\frac{i}{\hbar}E_n^{(0)}t} + \sum_k a_{kn}^{'(1)}(t)|\Psi_k^{(0)}\rangle e^{\frac{i}{\hbar}E_k^{(0)}t}.$$

As $\hat{V}(t)$ approaches constant value, the second term in the last expression coincides with the correction to the wave function coming from the time-independent perturbation theory. This correction determines the shift of the original wave functions due to small (constant) perturbation. Therefore, it does not contribute to the transition probability, which is determined by $a_{kn}^{'(1)}(t)$. Taking square, we have for excitation probability

$$w_{n0} = \frac{1}{\hbar^2\omega_{n0}^2}\left|\int\frac{\partial V_{n0}}{\partial t}e^{i\omega_{n0}t}dt\right|^2.$$

Now, we make use of the word "sudden". This means that the phase factor $e^{i\omega_{n0}t}$ varies slowly compared to $V_{n0}^{'}$, and one can put it in front of the integral, estimated at time $t = t_0$. Calculation of the remnant leads to the final answer

$$w_{n0} = \frac{|V_{n0}|^2}{\hbar^2 \omega_{n0}^2}.$$

Exercice 3 *Atom's jolt - calculation of the last integral*

Recall that the full probability for atom to excite is

$$P = 1 - w_{00},$$

where

$$w_{00} = \left| \frac{2}{a^3} \int_0^\infty r^2 dr \int_{-1}^{1} d\cos\theta e^{-2r/a} e^{-iqr\cos\theta} \right|^2 = |I|^2.$$

Our goal is to calculate the integral I. Integration over angular variables gives

$$I = \frac{4i}{a^3} \int_0^\infty e^{-2r/a} r^2 \frac{-i}{qr} \frac{e^{iqr} - e^{-iqr}}{2i} dr = \frac{4}{qa^3} \int_0^\infty r \sin(qr) e^{-2r/a} dr.$$

The last integral can be handled by integration by parts. Define

$$J_1 = \int_0^\infty \cos(qr) e^{-2r/a} dr,$$

$$J_2 = \int_0^\infty \sin(qr) e^{-2r/a} dr.$$

Then,

$$\int_0^\infty r \sin(qr) e^{-2r/a} dr = -\frac{1}{q} \int_0^\infty r e^{-2r/a} d\cos(qr) = \frac{J_1}{q} - \frac{2}{q^2 a} \int_0^\infty r e^{-2r/a} d\sin(qr) =$$

$$= \frac{J_1}{q} + \frac{2J_2}{q^2 a} - \frac{4}{q^2 a^2} \int r \sin(qr) e^{-2r/a} dr.$$

That is,

$$\int_0^\infty r \sin(qr) e^{-2r/a} dr = \left(1 + \frac{4}{q^2 a^2}\right)^{-1} \left(\frac{J_1}{q} + \frac{2J_2}{q^2 a}\right). \tag{1}$$

In the same manner we find J_1 and J_2:

$$J_1 = \frac{1}{q} \int_0^\infty e^{-2r/a} d\sin(qr) = -\frac{2}{q^2 a} \int_0^\infty e^{-2r/a} d\cos(qr) =$$

$$= \frac{2}{q^2 a} - \frac{4J_1}{q^2 a^2}.$$

Thus,

$$J_1 = \frac{2a}{q^2 a^2 + 4},$$

and we notice that

$$J_2 = \frac{qa}{2} J_1.$$

Substituting this into Eq.(1), we have for I,

$$I = \frac{4}{qa^3}\left(1 + \frac{4}{q^2a^2}\right)^{-1} \frac{4}{q^2a}\frac{qa^2}{q^2a^2+4} = \frac{1}{\left(1+\frac{q^2a^2}{4}\right)^2}.$$

The answer is then

$$P = 1 - \frac{1}{\left(1+\frac{q^2a^2}{4}\right)^4}.$$