$$\left[\hat{a},\hat{a}^{\dagger}\right]=1$$
$$\left[\hat{a}^{\dagger}\hat{a},\hat{a}\right]=\hat{a}^{\dagger}\hat{a}\hat{a}-\left(1+\hat{a}^{\dagger}\hat{a}\right)\hat{a}=-\hat{a}$$
$$\left[\hat{a}^{\dagger}\hat{a},\left[\hat{a}^{\dagger}\hat{a},\hat{a}\right]\right]=-\left[\hat{a}^{\dagger}\hat{a},\hat{a}\right]=+\hat{a}$$
$$\left[\hat{a}^{\dagger}\hat{a},...\left[\hat{a}^{\dagger}\hat{a},\hat{a}\right]...\right]=\left(-1\right)^{n}\hat{a}$$
$$\hat{U}=e^{-i\omega\hat{a}^{\dagger}\hat{a}t}$$
$$\hat{U}^{\dagger}\hat{a}\hat{U}=\hat{a}-\frac{i\omega}{1!}\left[\hat{a}^{\dagger}\hat{a},\hat{a}\right]+\frac{\omega^{2}}{2!}\left[\hat{a}^{\dagger}\hat{a},\left[\hat{a}^{\dagger}\hat{a},\hat{a}\right]\right]+...+\frac{\left(-i\omega\right)^{n}}{n!}\left[\hat{a}^{\dagger}\hat{a},\left[\hat{a}^{\dagger}\hat{a},...\left[\hat{a}^{\dagger}\hat{a},\hat{a}\right]...\right]\right]$$
$$\hat{U}^{\dagger}\hat{a}\hat{U}=\hat{a}+\frac{i\omega t}{1!}\hat{a}+\frac{\left(\omega t\right)^{2}}{2!}\hat{a}+...+\frac{\left(i\omega t\right)^{n}}{n!}\hat{a}=e^{i\omega t}\hat{a}$$
$$\hat{U}^{\dagger}\hat{a}\hat{U}=e^{+i\omega t}\hat{a}$$
$$\hat{U}^{\dagger}\hat{a}^{\dagger}\hat{U}=e^{-i\omega t}\hat{a}^{\dagger}$$
$$\hat{S}^{\dagger}\hat{U}^{\dagger}\hat{a}\hat{U}\hat{S}=e^{+i\omega t}\hat{S}^{\dagger}\hat{a}\hat{S}$$
$$\hat{S}^{\dagger}\hat{U}^{\dagger}\hat{a}^{\dagger}\hat{U}\hat{S}=e^{-i\omega t}\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S}$$
$$\hat{S}^{\dagger}\hat{a}\hat{S}=\cosh\left(r\right)\hat{a}-e^{+i\theta}\sinh\left(r\right)\hat{a}^{\dagger}$$
$$\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S}=\cosh\left(r\right)\hat{a}^{\dagger}-e^{-i\theta}\sinh\left(r\right)\hat{a}$$
$$\left\langle 0\right|\hat{S}^{\dagger}\hat{a}\hat{S}\left|0\right\rangle=0=\left\langle 0\right|\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S}\left|0\right\rangle$$
$$\left\langle 0\right|\hat{S}^{\dagger}\hat{a}^{2}\hat{S}\left|0\right\rangle=\left\langle 0\right|\hat{S}^{\dagger}\hat{a}\hat{S}\hat{S}^{\dagger}\hat{a}\hat{S}\left|0\right\rangle=-e^{+i\theta}\cosh\left(r\right)\sinh\left(r\right)$$
$$\left\langle 0\right|\hat{S}^{\dagger}\hat{a}^{2\dagger}\hat{S}\left|0\right\rangle=\left\langle 0\right|\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S}\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S}\left|0\right\rangle=-e^{-i\theta}\cosh\left(r\right)\sinh\left(r\right)$$
$$\left\langle 0\right|\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{S}\left|0\right\rangle=\left\langle 0\right|\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S}\hat{S}^{\dagger}\hat{a}\hat{S}\left|0\right\rangle=\cosh^{2}\left(r\right)$$
$$\left\langle 0\right|\hat{S}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{S}\left|0\right\rangle=\left\langle 0\right|\hat{S}^{\dagger}\hat{a}\hat{S}\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S}\left|0\right\rangle=\sinh^{2}\left(r\right)$$
$$\hat{U}^{\dagger}\hat{a}^{2}\hat{U}=\hat{U}^{\dagger}\hat{a}\hat{U}\hat{U}^{\dagger}\hat{a}\hat{U}=e^{+2i\omega t}\hat{a}^{2}$$
$$\hat{U}^{\dagger}\hat{a}^{2}\hat{U}=\hat{U}^{\dagger}\hat{a}\hat{U}\hat{U}^{\dagger}\hat{a}\hat{U}=e^{-2i\omega t}\hat{a}^{2\dagger}$$
$$\hat{U}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{U}=\hat{U}^{\dagger}\hat{a}^{\dagger}\hat{U}\hat{U}^{\dagger}\hat{a}\hat{U}=\hat{a}^{\dagger}\hat{a}$$
$$\hat{U}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{U}=\hat{U}^{\dagger}\hat{a}\hat{U}\hat{U}^{\dagger}\hat{a}^{\dagger}\hat{U}=\hat{a}\hat{a}^{\dagger}$$
$$\left\langle 0\right|\hat{S}^{\dagger}\hat{U}^{\dagger}\hat{a}^{2}\hat{U}\hat{S}\left|0\right\rangle\left(t\right)=-e^{+i\theta}\cosh\left(r\right)\sinh\left(r\right)e^{+2i\omega t}$$
$$\left\langle 0\right|\hat{S}^{\dagger}\hat{U}^{\dagger}\hat{a}^{2\dagger}\hat{U}\hat{S}\left|0\right\rangle\left(t\right)=-e^{-i\theta}\cosh\left(r\right)\sinh\left(r\right)e^{-2i\omega t}$$
$$\left\langle 0\right|\hat{S}^{\dagger}\hat{U}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{U}\hat{S}\left|0\right\rangle\left(t\right)=\cosh^{2}\left(r\right)$$
$$\left\langle 0\right|\hat{S}^{\dagger}\hat{U}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{U}\hat{S}\left|0\right\rangle\left(t\right)=\sinh^{2}\left(r\right)$$
$$\hat{x}=\hat{a}+\hat{a}^{\dagger}\Rightarrow\hat{x}^{2}=\hat{a}^{2\dagger}+\hat{a}^{2}+\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger}$$
$$\hat{p}=i\left(\hat{a}-\hat{a}^{\dagger}\right)\Rightarrow\hat{p}^{2}=-\left(\hat{a}^{2\dagger}+\hat{a}^{2}-\hat{a}^{\dagger}\hat{a}-\hat{a}\hat{a}^{\dagger}\right)=-\hat{a}^{2\dagger}-\hat{a}^{2}+\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger}$$
$$\cosh^{2}\left(r\right)-\sinh^{2}\left(r\right)=1\Rightarrow\sinh^{2}\left(r\right)=\cosh^{2}\left(r\right)-1\Rightarrow\cosh^{2}\left(r\right)+\sinh^{2}\left(r\right)=2\cosh^{2}\left(r\right)-1$$
$$\left\langle\psi\right|\hat{x}^{2}\left|\psi\right\rangle\left(t\right)=\left\langle\psi\right|\hat{a}^{2\dagger}+\hat{a}^{2}+\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger}\left|\psi\right\rangle=-2\cosh\left(r\right)\sinh\left(r\right)\cos\left(\theta+2\omega t\right)+\cosh\left(2r\right)$$
$$\left\langle\psi\right|\hat{p}^{2}\left|\psi\right\rangle\left(t\right)=\left\langle\psi\right|-\hat{a}^{2\dagger}-\hat{a}^{2}-\hat{a}^{\dagger}\hat{a}-\hat{a}\hat{a}^{\dagger}\left|\psi\right\rangle=2\cosh\left(r\right)\sinh\left(r\right)\cos\left(\theta+2\omega t\right)+\cosh\left(2r\right)$$
$$\Delta\hat{x}=\sqrt{\cosh\left(2r\right)-2\cosh\left(r\right)\sinh\left(r\right)\cos\left(\theta+2\omega t\right)}$$
$$\Delta\hat{p}=\sqrt{\cosh\left(2r\right)+2\cosh\left(r\right)\sinh\left(r\right)\cos\left(\theta+2\omega t\right)}$$