# ASSIGNMENT #1
# (SOLUTIONS)

## Q NO 1

The gas in question is a closed, thermally isolated system, described by the microcanonical ensemble.

⇒ Entropy is defined as

$$S = k \ln \Gamma(E)$$

where $\Gamma(E)$ is the number of microstates corresponding to the fixed energy $E$.

⇒ The Hamiltonian of the given ideal gas of $N$ classical monatonic particles, contained in a $d$-dimensional box of volume $V$; is

$$H = \sum_{i=1}^{dN} \frac{p_i^2}{2m}$$

and $\Gamma(E)$ is

$$\Gamma(E) = \frac{1}{h^{dN}} \int_{H(q,p)=E} dq^{dN}\, dp^{dN}$$

$$= \frac{V^N}{h^{dN}} \int_{\sum^{dN} p_i^2 = 2mE} dp^{dN}$$

where the factor $V^N$ is the result of integrating over the coordinates $q$'s - and momentum integral is the volume of a $dN$- dimensional sphere with radius $R = \sqrt{2mE}$.

$$\Gamma(E) = \frac{V^N}{h^{dN}} \Omega_{dN}(R) \quad \text{———} \ ①$$

⇒ Obviously, $\Omega_{dN}(R)$ will be propotional to $R^{dN}$, so let it be written as.

$$\Omega_{dN} = \int\limits_{x_1^2 + x_2^2 + \cdots x_{dN}^2 = R^2} dx_1 \, dx_2 \cdots\cdots dx_{dN} = C_{dN} R^{dN} \quad \text{———} \ ②$$

where $C_{dN}$ is a constant which depends only on the dimensionality of space. Clearly, the "volume element" $d\Omega_{dN}(R)$ can also be written as

$$d\Omega_{dN} = dx_1 \, dx_2 \cdots dx_{dN} = d(C_{dN} R^{dN})$$
$$= dN \, C_{dN} R^{dN-1} dR$$

⇒ To evaluate $C_{dN}$, we make use of the formula

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

⇒ Multiplying $(n = dN)$ such integrals, one for each $x_i$, we obtain

$$\left(\pi^{\frac{1}{2}}\right)^{dN} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots\cdots \int_{-\infty}^{\infty} dx_{dN} \, e^{-(x_1^2 + x_2^2 + \cdots x_{dN}^2)}$$

$$\pi^{dN/2} = \int_0^\infty e^{-R^2} dN\, C_{dN} R^{dN-1} dR$$

$$\Rightarrow \frac{\pi^{dN/2}}{C_{dN}\, dN} = \int_0^\infty e^{-R^2} R^{dN-1} dR \quad \text{———} \quad Ⓐ$$

∵ Gaussian integrals $\displaystyle\int_0^\infty x^n e^{-ax^2} dx = \frac{\frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{a^{n+1}}}$

therefore, Eq Ⓐ becomes

$$\frac{\pi^{dN/2}}{C_{dN}\, dN} = \frac{1}{2} \frac{\Gamma\left(\frac{dN+1-1}{2}\right)}{1}$$

$$\Rightarrow \frac{\pi^{dN/2}}{\frac{dN}{2}} \cdot \frac{1}{\Gamma\left(\frac{dN}{2}\right)} = C_{dN}$$

$$\because \Gamma(n) = (n-1)!$$

$$\Rightarrow \Gamma\left(\frac{dN}{2}\right) = \left(\frac{dN}{2}-1\right)!$$

$$\Rightarrow C_{dN} = \frac{\pi^{dN/2}}{\frac{dN}{2}\left(\frac{dN}{2}-1\right)!}$$

$$C_{dN} = \frac{\pi^{dN/2}}{\left(\frac{dN}{2}\right)!} \qquad \because n(n-1)! = n!$$

⇒ plug in Eq ②,

$$\Omega_{dN} = \frac{\pi^{dN/2}}{\left(\frac{dN}{2}\right)!} R^{dN}$$

and relation for ↓ accessible states is

$$\Gamma(E) = \frac{V^N}{h^{dN}} \frac{\pi^{dN/2} R^{dN}}{\left(\frac{dN}{2}\right)!}$$

$$\Rightarrow \quad S = k \ln \Gamma(E)$$

$$= k \ln\left[\frac{V^N \pi^{dN/2} R^{dN}}{h^{dN} \left(\frac{dN}{2}\right)!}\right]$$

$$\because \ln N! \simeq N \ln N - N \quad \text{for } N \gg 1$$

$$\text{and} \quad R = \sqrt{2mE}$$

$$\Rightarrow \quad S = k \ln\left[\frac{V^N \pi^{dN/2} (2mE)^{dN/2}}{h^{dN} \left(\frac{dN}{2}\right)!}\right]$$

$$\simeq k\left[\ln \pi^{dN/2} - \frac{dN}{2}\ln\left(\frac{dN}{2}\right) + \frac{dN}{2}\right] + k \ln\left[\frac{V^N (2mE)^{dN/2}}{h^{dN}}\right]$$

$$= k\left[N \ln \pi^{d/2} - N\frac{d}{2}\ln\left(\frac{dN}{2}\right) + N\frac{d}{2}\right] + kN \ln\left(\frac{V (2mE)^{d/2}}{h^d}\right)$$

$$= kN\left[\ln \pi^{d/2} - \frac{d}{2}\ln\left(\frac{dN}{2}\right) + \frac{d}{2} + \ln\left(\frac{V (2mE)^{d/2}}{h^d}\right)\right]$$

$$= kN\left[\ln \pi^{d/2} - \ln\left(\frac{dN}{2}\right)^{d/2} + \ln V + \ln\left(\frac{2mE}{h^2}\right)^{d/2} + \frac{d}{2}\right]$$

$$S \simeq kN\left\{\ln\left[V\left(\frac{4\pi mE}{dNh^2}\right)^{d/2}\right] + \frac{d}{2}\right\} \quad \text{———} \; ③$$

⇒ To obtain equation of state, use the First law of thermodynamics

$$\because \quad dE = dQ + dW \qquad (\text{for } N = \text{constant})$$

$$dE = TdS - PdV$$

$$\Rightarrow \quad dS = \frac{1}{T}dE + \frac{P}{T}dV \text{ ——— } Ⓑ$$

$$\Rightarrow \quad S = S(E,V)$$

$$dS = \left(\frac{\partial S}{\partial E}\right)_V dE + \left(\frac{\partial S}{\partial V}\right)_E dV \text{ ——— } Ⓒ$$

$$\Rightarrow \quad \frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_E$$

$$P = T\left(\frac{\partial S}{\partial V}\right)_E \text{ ——— } ④$$

Take derivative of Eq ③ w.r.t V at constant E, N

$$\Rightarrow \quad \left(\frac{\partial S}{\partial V}\right)_{E,N} = kN\frac{\partial}{\partial V}(\ln V) = \frac{kN}{V} \text{ — } ⑤$$

⇒ From Eq ④ and Eq ⑤;

$$P = T\frac{kN}{V} \quad \Rightarrow \quad PV = NKT$$

⇒ We can also find the temperature:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} \qquad \because \text{compare Eq Ⓑ and Ⓒ}$$

→ take derivative of Eq ③ w.r.t E at constant V, N

$$\left(\frac{\partial S}{\partial E}\right)_{V,N} = kN\frac{\partial}{\partial E}\left(\ln E^{d/2}\right) = kN\frac{\partial}{\partial E}\left(\frac{d}{2}\ln E\right)$$

$$= kN\frac{d}{2}\frac{\partial}{\partial E}(\ln E) = \frac{dkN}{2E}$$

$$\Rightarrow \quad \frac{1}{T} = \frac{dNk}{2E}$$

or $E = \frac{d}{}NKT$ is relation for energy of a monatomic classical ideal gas.

**QNo2**

According to the relativistic energy-momentum relation,

$$E = \sqrt{P^2C^2 + m^2C^4},$$

while for highly relativistic particles, their rest energy can be neglected, and the total energy of the gas is

$$H = c\sum_{i=1}^{N} P_i$$

so D is the region $\sum_i P_i = \frac{E}{c}$ and $\Gamma(E)$ is the volume of phase space corresponding to energy equal to E.

$$\Gamma(E) = \frac{1}{h^{3N}}\int_D \prod_{i=1}^{N} d^3q_i\, d^3P_i$$

⇒ Since the positions and the directions of the momenta of the particles are unconstrained, we can write for each particle; $\int d^3q \int d^3P = 4\pi V \int P^2 dP$, so

$$\Gamma(E) = \left(\frac{4\pi V}{h^3}\right)^N \int_D \prod_{i=1}^{N} \int P_i^2\, dP_i \equiv \underline{\left(\frac{4\pi V}{h^3}\right)^N I_N(E)} \quad \textcircled{\times}$$

⇒ To evaluate integral $I_N(E)$, we note that, if the momentum of one particle is fixed to the value P, then the integral for the remaining particles is $I_{N-1}(E-PC)$

⇒ We have

$$I_N(E) = \int_0^{E/C} dP\, P^2\, I_{N-1}(E-PC)$$

Then dimension analysis tells us that

$$I_N = C_N \left(\frac{E}{c}\right)^{3N}$$

$$\because \int_0^P P^2 dP = \frac{P^3}{3}$$

and substituting this in the above equation yields a recussion relation for the number $C_N$:

$$C_N \left(\frac{E}{c}\right)^{3N} = \int_0^{E/c} P^2 dP \; C_{N-1} \left(\frac{E}{c} - P\right)^{3N-3}$$

$$\because \; I_{N-1} = C_{N-1} \left(\frac{E}{c}\right)^{3(N-1)}$$

$$\Rightarrow I_{N-1}(E - Pc) = C_{N-1} \left(\frac{E}{c} - P\right)^{3N-3}$$

$$\Rightarrow C_N = \left(\frac{c}{E}\right)^{3N} \int_0^{E/c} P^2 dP \; C_{N-1} \left(\frac{E}{c} - P\right)^{3N-3}$$

$$= \left(\frac{c}{E}\right)^{3N-3+3} \int_0^{E/c} P^2 dP \; C_{N-1} \left(\frac{E}{c} - P\right)^{3N-3}$$

$$= \left(\frac{c}{E}\right)^{3} \int_0^{E/c} P^2 dP \; C_{N-1} \left(1 - \frac{PE}{E}\right)^{3N-3}$$

Let $x = \frac{Pc}{E}$

$$\Rightarrow dx = dP \, \frac{c}{E},$$

$$= \int_0^{E/c} \left(\frac{cP}{E}\right)^2 \left(\frac{c}{E} dP\right) C_{N-1} \left(1 - \frac{Pc}{E}\right)^{3N-3}$$

$$= \int_0^1 x^2 dx \, (1-x)^{3N-3} \, C_{N-1}$$

as $P \to 0$, $x \to 0$

$P \to \frac{E}{c}$, $x \to 1$

$$\Rightarrow \frac{C_N}{C_{N-1}} = \int_0^1 x^2 (1-x)^{3N-3} dx$$

Let $\int_0^1 x^2(1-x)^{3N-3}\,dx$

$\because 1-x = z$

$1-z = x$

$-dz = dx$

as $x \to 0,\ z \to 1$

$x \to 1,\ z = 0$

$$= -\int_1^0 (1-z)^2 z^{3N-3}\,dz$$

$$= \int_0^1 (1-z)^2 z^{3N-3}\,dz$$

$$= \int_0^1 (1+z^2-2z) z^{3N-3}\,dz$$

$$= \int_0^1 z^{3N-3}\,dz + \int_0^1 z^{2+3N-3}\,dz - 2\int_0^1 z^{1+3N-3}\,dz$$

$$= \left.\frac{z^{3N-3+1}}{(3N-3+1)}\right|_0^1 + \left.\frac{z^{3N-1+1}}{(3N-1+1)}\right|_0^1 - 2\left.\frac{z^{3N-2+1}}{(3N-2+1)}\right|_0^1$$

$$= \frac{1}{3N-2} + \frac{1}{3N} - \frac{2}{3N-1}$$

$$= \frac{3N(3N-1)+(3N-2)(3N-1)-2(3N)(3N-2)}{3N(3N-1)(3N-2)}$$

$$= \frac{3N(\cancel{3N}) - \cancel{3N} + 3\cancel{N}(3N) - \cancel{3N} - 2(\cancel{3N}) + 2 - 2(\cancel{3N})^2 + 4(\cancel{3N})}{3N(3N-1)(3N-2)}$$

$$= \frac{2}{3N(3N-1)(3N-2)}$$

$$= \frac{2\,(3N-3)(3N-4)(3N-5)\cdots\cdots 1}{3N\,(3N-1)(3N-2)(3N-3)(3N-4)(3N-5)\cdots\cdots 1}$$

$$= \frac{2(3N-3)!}{(3N)!}$$

$\because k! = k(k-1)(k-2)\cdots 1$

$$\Rightarrow C_N = \frac{2(3N-3)!}{(3N)!} C_{N-1}$$

$\Rightarrow$ Clearly, we have $C_1 = \int_0^1 x^2(1-x)^0 dx \; C_0$

$$= \int_0^1 x^2 dx = \frac{1}{3}$$

and the solution to the recursion relation is

$$C_N = \frac{2^N}{(3N)!}$$

$$\Rightarrow \quad I_N(E) = C_N \left(\frac{E}{c}\right)^{3N} = \frac{2^N}{(3N)!}\left(\frac{E}{c}\right)^{3N}$$

$\Rightarrow$ Therefore, Eq Ⓧ for volume of phase space becomes

$$\Gamma(E) = \left(\frac{4\pi V}{h^3}\right)^N \frac{2^N}{(3N)!}\left(\frac{E}{c}\right)^{3N}$$

$$\Gamma(E) = \left(\frac{V 8\pi E^3}{h^3 c^3}\right)^N \frac{1}{(3N)!}$$

$\Rightarrow$ For indistinguisible particles,

$$\Gamma(E) = \frac{1}{N!}\frac{1}{(3N)!}\left(\frac{8\pi V E^3}{h^3 C^3}\right)^N$$

$$S = k_B \ln \Gamma(E)$$

$$= k_B \ln\left(\frac{1}{N!}\left(\frac{8\pi V E^3}{h^3 C^3}\right)^N \frac{1}{(3N)!}\right) \quad \text{——} \; ⓨ$$

⇒ Temperature is defined as

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N}$$

∵ $$S = k_B \ln\left(\frac{1\,(8\pi V)^N (hc)^{-3N}}{(N!)(3N!)} \cdot E^{3N}\right)$$

$$\Rightarrow \left(\frac{\partial S}{\partial E}\right)_{V,N} = 3Nk_B \frac{\partial}{\partial E}(\ln E)$$

$$\frac{1}{T} = \frac{3Nk_B}{E}$$

$$\boxed{\frac{E}{3Nk_B} = T}$$ and

$$\because P = T\left(\frac{\partial S}{\partial V}\right)_{E,N}$$

$$= T\frac{\partial}{\partial V}\left(k_B \ln V^N\right) = NTk_B \frac{\partial}{\partial V}\ln V = \frac{Nk_BT}{V}$$

$$\Rightarrow \quad PV = Nk_BT.$$

The equation of state is thus the same as for a non-relativistic gas: $PV = Nk_BT$.

∵ Entropy relation in Eq Ⓨ

$$S = k_B \ln\left(\frac{1}{N!\,(3N)!}\left(\frac{8\pi V E^3}{h^3 c^3}\right)^N\right)$$

∵ for $N \gg 1$, $\ln N! \simeq N\ln N - N$

⇒ Similarly, $\ln 3N! \simeq 3N\ln 3N - 3N$

$$\Rightarrow S \simeq k_B\left[-N\ln N + N - 3N\ln 3N + 3N + N\ln\left(\frac{8\pi V E^3}{h^3 C^3}\right)\right]$$

$$= k_B\left[4N - N\ln N - N\ln(3N)^3 + N\ln\left(\frac{8\pi V E^3}{h^3 C^3}\right)\right]$$

$$= k_B\left[4N + N\ln\left(\frac{8\pi V E^3}{27N^4 h^3 C^3}\right)\right]$$

$$\Rightarrow S \simeq k_B N\left[\ln\left(\frac{8\pi V E^3}{27N^4 h^3 C^3}\right) + 4\right] \text{———} \textcircled{\dot{Z}}$$

**Specific heat ratio :-**

$\because \quad E = T3Nk_B$

$\Rightarrow \quad E^3 = T^3\, 3^3 N^3 k_B^3$ plug in $\textcircled{\dot{Z}}$

$$\Rightarrow S \simeq k_B N\ln\left(\frac{8\pi V\, 27N^3 k_B^3 T^3}{27N^4 h^3 C^3}\right)$$

$$S \simeq k_B N\ln\left(\frac{C_1 T^3 V}{N}\right) \qquad \text{where } C_1 = \frac{8\pi k_B^3}{h^3 C^3}$$

$$= k_B N\ln\left(\frac{C_1 T^3 N k_B T}{PN}\right) \qquad \because V = \frac{Nk_BT}{P}$$

$$= k_B N\ln\left(\frac{C_2 T^4}{P}\right) \qquad \because C_1 \text{ and } C_2 \text{ are constants.}$$

$$\Rightarrow S(T,V,N) = Nk_B\ln\left(\frac{C_1 T^3 V}{N}\right)$$

$$S(T,P,N) = Nk_B\ln\left(\frac{C_2 T^4}{P}\right)$$

$\Rightarrow$ The specific heats are then

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = 3\frac{T}{T^3}Nk_B T^2 = 3Nk_B$$

and $$C_p = T\left(\frac{\partial S}{\partial T}\right)_{P,N} = \frac{T}{T^4}4Nk_B T^3 = 4Nk_B$$

$$\Rightarrow \gamma = \frac{C_p}{C_V} = \frac{4Nk_B}{3Nk_B} = \frac{4}{3}.$$

## QNO3

Given that, a perfect crystal has N lattice sites and M interstitial locations. Total n number of atoms are displaced from site and place it in interstitial.

(a) The number of ways to choose n atoms to remove from N sites is

$$C_n^N = \frac{N!}{n!\,(N-n)!}$$

(b) The number of ways to choose n interstitials out of M is

$$C_n^M = \frac{M!}{n!\,(M-n)!}$$

(c) The number of states at total energy $(E = n\Delta)$ is the number of ways to choose n atoms from N sites and place them in M interstitials, that is

$$\Gamma\left(n=\frac{E}{\Delta}\right) = C_n^N \times C_n^M$$

$$= \frac{N!}{n!\,(N-n)!}\ \frac{M!}{n!\,(M-n)!}$$

$\Rightarrow$ entropy is

$$S(E=\Delta n) = k_B \ln \Gamma(E)$$

$$= k_B \ln\left(\frac{N!\,M!}{n!\,(N-n)!\;n!\,(M-n)!}\right)$$

$\Rightarrow$ Using Stirling's approximation; for $N \gg 1$ $\ln N! \simeq N\ln N - N$

$$S = k_B \left[ \ln N! + \ln M! - 2\ln n! - \ln (N-n)! - \ln (M-n)! \right]$$

$$\simeq k_B \left[ N\ln N \cancel{-N} + M\ln M \cancel{-M} - 2n\ln n + \cancel{2n} - (N-n)\ln(N-n) \cancel{+N-n} \right.$$
$$\left. - (M-n)\ln(M-n) + \cancel{M-n} \right]$$

$$= k_B \left[ N\ln N + M\ln M - 2n\ln n - (N-n)\ln(N-n) \underset{\textcircled{A}}{\underline{-(M-n)\ln(M-n)}} \right]$$

$$= k_B \left[ N\ln N + M\ln M - 2n\ln n - N\ln(N-n) + n\ln(N-n) - M\ln(M-n) \right.$$
$$\left. + n\ln(M-n) \right]$$

$$= k_B \left[ N\ln\left(\frac{N}{N-n}\right) + M\ln\left(\frac{M}{M-n}\right) + n\ln\left(\frac{N-n}{n}\right) + n\ln\left(\frac{M-n}{n}\right) \right]$$

$$= k_B \left[ -N\ln\left(\frac{N-n}{N}\right) - M\ln\left(\frac{M-n}{M}\right) + n\ln\left(\frac{N-n}{n}\right) - n\ln N + n\ln N + \right.$$
$$\left. n\ln\left(\frac{M-n}{n}\right) - n\ln M + n\ln M \right]$$

$$= k_B \left[ -N\ln\left(\frac{N-n}{N}\right) - M\ln\left(\frac{M-n}{M}\right) + n\ln\left(\frac{N-n}{Nn}\right) + n\ln\left(\frac{M-n}{nM}\right) \right.$$
$$\left. + n\ln NM \right]$$

$$= k_B \left[ -N\ln\left(\frac{N-n}{N}\right) - M\ln\left(\frac{M-n}{M}\right) + n\ln\left(\frac{N-n}{N}\right) + n\ln\left(\frac{M-n}{M}\right) + n\ln\left(\frac{NM}{n^2}\right) \right]$$

$$= k_B \left[ n\ln\left(\frac{NM}{n^2}\right) - (N-n)\ln\left(\frac{N-n}{N}\right) - (M-n)\ln\left(\frac{M-n}{M}\right) \right]$$

$$S \simeq k_B \left[ n\ln\left(\frac{NM}{n^2}\right) - (N-n)\ln\left(1 - \frac{n}{N}\right) - (M-n)\ln\left(1 - \frac{n}{M}\right) \right]$$

⇒ Temperature is defined as

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} \qquad \text{where} \quad E = \Delta n$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial \Delta n}\right) = \frac{\partial k_B \ln \Gamma(n)}{\partial (\Delta n)}$$

$$\Rightarrow \frac{1}{T} = \frac{k_B}{\Delta} \frac{\partial \ln \Gamma(n)}{\partial n}$$

$$\Rightarrow \frac{\Delta}{k_B T} = \frac{\partial}{\partial n} \ln(\Gamma(n))$$

⇒ From Eq Ⓐ,

$$\ln \Gamma(n) = N\ln N + M\ln M - 2n\ln n - (N-n)\ln(N-n) - (M-n)\ln(M-n)$$

$$\Rightarrow \frac{\partial}{\partial n} \ln \Gamma(n) = \frac{\partial}{\partial n}\left[-2n\ln n - (N-n)\ln(N-n) - (M-n)\ln(M-n)\right]$$

$$= -2\ln n - 2 - \left[\ln(N-n)\cdot(-1) - 1 + \ln(M-n)\cdot(-1) - 1\right]$$

$$= -2\ln n - \cancel{2} + \ln(N-n) \cancel{+1} + \ln(M-n) \cancel{+1}$$

$$= \ln\left(\frac{(N-n)(M-n)}{n^2}\right)$$

$$\Rightarrow \frac{\Delta}{k_B T} = \ln\left(\frac{(N-n)(M-n)}{n^2}\right)$$

## Q NO4

To find the Helmholtz free energy $F(T,N)$ for the given system in microcanonical ensemble,

$$F(T,N) = E(T,N) - TS(E,N) \quad\text{———}\ ⑤$$

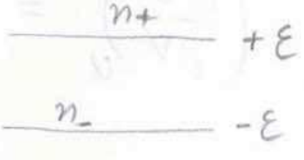
⇒ Firstly, we have to find entropy and plugging '$S$' in above relation, simplying it for $F(T,N)$

⇒ According to given condition,

$$N = \text{total number of particles} = n_+ + n_-$$

where $n_+$ is number of particles have energy $+\varepsilon$
$n_-$ is " " " " " $-\varepsilon$

$$\Rightarrow \quad E_{total} = n_+\varepsilon_+ + n_-\varepsilon_-$$
$$= (n_+ - n_-)\varepsilon$$



$\Rightarrow$ Using binomial distribution, total number of accessible microstates are

$$\Omega(E,N) = \frac{N!}{n_+!\, n_-!}$$
$$= \frac{N!}{n_+!\,(N-n_+)!}$$

and entropy is

$$S(E,N) = k \ln \Omega(E,N)$$
$$= k\left[\ln N! - \ln n_+! - \ln (N-n_+)!\right]$$

$\Rightarrow$ Using Stirling approximation,

$$\ln N! \simeq N\ln N - N$$

$$\Rightarrow S(E,N) \simeq k\left[N\ln N - N - n_+\ln n_+ + n_+ - (N-n_+)\ln(N-n_+) + (N-n_+)\right]$$

$$= k\left[N\ln N - \cancel{(N-n_+)} + \cancel{(N-n_+)} - n_+\ln n_+ - (N-n_+)\ln(N-n_+)\right] \quad ⑤$$

$$= k\left[N\ln N - n_+\ln n_+ - (N-n_+)\ln(N-n_+)\right]$$

$$= k\left[N\ln N - n_+\ln n_+ - N\ln(N-n_+) + n_+\ln(N-n_+)\right]$$

$$= k\left[N\ln\left(\frac{N}{N-n_+}\right) + n_+\ln\left(\frac{N-n_+}{n_+}\right)\right]$$

$$= kN\left[\ln\left(\frac{N}{N-n_+}\right) + N^{-1}n_+\ln\left(\frac{N-n_+}{n_+}\right)\right]$$

assume $x = \frac{n_+}{N}$ , $y = \frac{n_-}{N}$

$$\Rightarrow \quad E = (n_+ - n_-)\varepsilon$$

$$\frac{E}{\varepsilon} = n_+ - (N-n_+)$$

$$\frac{E}{\varepsilon} = 2n_+ - N$$

$$\Rightarrow \left(\frac{E}{\varepsilon N} + 1\right)\frac{1}{2} = \frac{n_+}{N}$$

$$x = \frac{1}{2}\left(\frac{E}{\varepsilon N} + 1\right) \quad — ⑥$$

Thus, entropy $S(E,N)$ is

$$\boxed{S \simeq -kN\left[(1-x)\ln(1-x) + x\ln x\right]} \quad — ⑦$$

⇒ To formalize Energy in terms of temperature T and N : E(T,N), firstly we have to find temperature.

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N \text{ --- } ⑧$$

$$\because \frac{\partial}{\partial E} = \frac{\partial X}{\partial E} \cdot \frac{\partial}{\partial X}$$

$$\because X = \frac{1}{2}\left(1 + \frac{E}{\varepsilon N}\right)$$

$$\frac{\partial X}{\partial E} = \frac{1}{2N\varepsilon}$$

$$\Rightarrow \frac{\partial}{\partial E} = \frac{1}{2N\varepsilon}\frac{\partial}{\partial X}$$

$$\Rightarrow \quad \frac{1}{T} = \frac{1}{2N\varepsilon}\left(\frac{\partial S}{\partial X}\right)_N \qquad \because \text{using Eq ⑦}.$$

$$\frac{1}{T} = \frac{-k}{2\varepsilon}\left[-\ln(1-x) - 1 + \ln x + 1\right]$$

$$= \frac{-k}{2\varepsilon}\left[\ln x - \ln(1-x)\right] = \frac{k}{2\varepsilon}\ln\left(\frac{1-x}{x}\right)$$

$$\Rightarrow \quad \frac{2\varepsilon}{kT} = \ln\left(\frac{1-x}{x}\right)$$

$$e^{2\varepsilon/kT} = \frac{1-x}{x}$$

$$\Rightarrow \ x = \left(1+e^{2\varepsilon/kT}\right)^{-1}$$

$$\frac{n_+}{N} = \frac{1}{2}\left(1+\frac{E}{N\varepsilon}\right) = x = \left(1+e^{2\varepsilon/kT}\right)^{-1}$$

$$\Rightarrow \ 1+\frac{E}{N\varepsilon} = 2\left(1+e^{2\varepsilon/kT}\right)^{-1}$$

$$\frac{E}{N\varepsilon} = -1 + \frac{2}{\left(1+e^{2\varepsilon/kT}\right)}$$

$$= \frac{-1-e^{2\varepsilon/kT}+2}{\left(1+e^{2\varepsilon/kT}\right)}$$

$$= \frac{\left(1-e^{2\varepsilon/kT}\right)}{\left(1+e^{2\varepsilon/kT}\right)}$$

$$\frac{E}{N\varepsilon} = -\left(\frac{e^{2\varepsilon/kT}-1}{e^{2\varepsilon/kT}+1}\right)$$

$$\boxed{E = -N\varepsilon \tanh\left(\frac{\varepsilon}{kT}\right)} \quad ⑨$$

⇒ To further simplify Eq ⑦ entropy, plug in x.

$$S \simeq -Nk\left[x\ln x + (1-x)\ln(1-x)\right]$$

$$= -Nk\left[\left(1+e^{2\varepsilon/kT}\right)^{-1}\ln\left(1+e^{2\varepsilon/kT}\right)^{-1} + \left(1-\left(1+e^{2\varepsilon/kT}\right)^{-1}\right)\cdot \ln\left(1-\left(1+e^{2\varepsilon/kT}\right)^{-1}\right)\right] \quad ⑩$$

⇒ Let

$$\left(1+e^{2\varepsilon/kT}\right)^{-1} = \frac{e^{-2\varepsilon/kT}}{1+e^{-2\varepsilon/kT}}$$

$$= \frac{e^{-y}}{1+e^{-y}} \qquad \because \frac{2\varepsilon}{kT} = y$$

plug in ⑩

$$\Rightarrow S \simeq -Nk\left[\left(\frac{e^{-y}}{1+e^{-y}}\right)\ln\left(\frac{e^{-y}}{1+e^{-y}}\right) + \left(1-\frac{e^{-y}}{1+e^{-y}}\right)\cdot \ln\left(1-\frac{e^{-y}}{1+e^{-y}}\right)\right].$$

$$S \simeq Nk\left[\left(\frac{ye^{-y}}{1+e^{-y}}\right) + \ln\left(1+e^{-y}\right)\right] \quad ⑪$$

⇒ By putting S from ⑪, E from ⑨ in ⑤, F(T, N) is

$$F(T,N) = E - TS$$

$$= -N\varepsilon \tanh(y/2) - TNk\left[\left(\frac{y e^{-y}}{1+e^{-y}}\right) + \ln\left(1+e^{-y}\right)\right]$$

$$\because \tanh(y/2) = \frac{1-e^{-y}}{1+e^{-y}}$$

$$\Rightarrow F(T,N) = -N\varepsilon\left(\frac{1-e^{-y}}{1+e^{-y}}\right) - kTN\left[\left(\frac{y e^{-y}}{1+e^{-y}}\right) + \ln\left(1+e^{-y}\right)\right]$$

$$= -N\varepsilon\left(\frac{1-e^{-y}}{1+e^{-y}}\right) - N\varepsilon 2\,\frac{e^{-y}}{1+e^{-y}} - kT\ln\left(1+e^{-y}\right)$$

$$\because y = \frac{2\varepsilon}{kT}$$

$$\Rightarrow \boxed{F(T,N)_{\text{Microcanonical}} \simeq -N\varepsilon - kTN\ln\left(1+e^{-2\varepsilon/kT}\right)}$$

# Assignment # 2

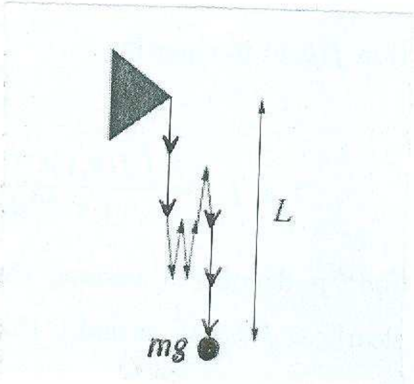
## Solutions

### QNo1

(a) The number of up-segments $N_\uparrow$ and the number of down-segments $N_\downarrow$ satisfy

$$N = N_\uparrow + N_\downarrow \quad \text{and}$$

$$L = a(N_\downarrow - N_\uparrow)$$



thus,

$$N_\uparrow = N - N_\downarrow \quad \text{—— (i)}$$

$$\because \quad L = aN_\downarrow - aN_\uparrow$$

$$L + aN_\uparrow = aN_\downarrow$$

$$\Rightarrow N_\downarrow = \frac{L}{a} + N_\uparrow \quad \text{plug in Eq (i)}$$

$$N_\uparrow = N - \left(\frac{L}{a} + N_\uparrow\right)$$

$$2N_\uparrow = N - \frac{L}{a} \quad \Rightarrow \quad \boxed{N_\uparrow = \frac{N}{2} - \frac{L}{2a}}$$

$\Rightarrow$ Similarly,

$$N_\downarrow = N - N_\uparrow$$

$$= N - \left(\frac{N}{2} - \frac{L}{2a}\right) = \frac{N}{2} + \frac{L}{2a}$$

$$\Rightarrow \quad \boxed{N_\downarrow = \frac{N}{2} + \frac{L}{2a}}$$

⇒ The total number of accessible microstates are

$$\Gamma(L) = \frac{N!}{N_{\uparrow}!\,N_{\downarrow}!}$$

$$\Gamma(L) = \frac{N!}{\left(\frac{N}{2}-\frac{L}{2a}\right)!\left(\frac{N}{2}+\frac{L}{2a}\right)!}$$

(b) Entropy; $S = k_B \ln \Gamma(L)$

$$S = k_B\left[\ln N! - \ln\left(\frac{N}{2}-\frac{L}{2a}\right)! - \ln\left(\frac{N}{2}+\frac{L}{2a}\right)!\right]$$

∵ Use Stirling approximation, for large N

$$\ln N! \simeq N\ln N - N$$

$$\Rightarrow S \simeq k_B\left[N\ln N - N - \left(\frac{N}{2}-\frac{L}{2a}\right)\ln\left(\frac{N}{2}-\frac{L}{2a}\right) + \left(\frac{N}{2}-\frac{L}{2a}\right) - \left(\frac{N}{2}+\frac{L}{2a}\right)\ln\left(\frac{N}{2}+\frac{L}{2a}\right) + \left(\frac{N}{2}+\frac{L}{2a}\right)\right]$$

$$= k_B\left[N\ln N - \left(\frac{N}{2}-\frac{L}{2a}\right)\ln\left(\frac{N}{2}-\frac{L}{2a}\right) - \left(\frac{N}{2}+\frac{L}{2a}\right)\ln\left(\frac{N}{2}+\frac{L}{2a}\right) - \cancel{N} + \cancel{\frac{N}{2}} - \cancel{\frac{L}{2a}} + \cancel{\frac{N}{2}} + \cancel{\frac{L}{2a}}\right]$$

$$= k_B\left[N\ln N - \left(\frac{N}{2}-\frac{L}{2a}\right)\ln\left(\frac{N}{2}-\frac{L}{2a}\right) - \left(\frac{N}{2}+\frac{L}{2a}\right)\cdot\ln\left(\frac{N}{2}+\frac{L}{2a}\right)\right]$$

$$= k_B\left[N\ln N - \left(\frac{N}{2}-\frac{L}{2a}\right)\ln\left[\frac{N}{2}\left(1-\frac{L}{Na}\right)\right] - \left(\frac{N}{2}+\frac{L}{2a}\right)\cdot\ln\left[\frac{N}{2}\left(1+\frac{L}{Na}\right)\right]\right]$$

$$= k_B \left[ N\ln N - \frac{N}{2}\ln\frac{N}{2} + \frac{L}{2a}\cancel{\ln\frac{N}{2}} - \left(\frac{N}{2} - \frac{L}{2a}\right)\ln\left(1 - \frac{L}{Na}\right) - \frac{N}{2}\ln\frac{N}{2} - \frac{L}{2a}\cancel{\ln\frac{N}{2}} - \left(\frac{N}{2} + \frac{L}{2a}\right)\ln\left(1 + \frac{L}{Na}\right) \right]$$

$$= k_B \left[ N\ln N - N\ln\frac{N}{2} - \left(\frac{N}{2} - \frac{L}{2a}\right)\ln\left(1 - \frac{L}{Na}\right) - \left(\frac{N}{2} + \frac{L}{2a}\right)\ln\left(1 + \frac{L}{Na}\right) \right]$$

$$= k_B \left[ N\cancel{\ln N} - N\cancel{\ln N} + N\ln 2 - \left(\frac{N}{2} - \frac{L}{2a}\right)\ln\left(1 - \frac{L}{Na}\right) - \left(\frac{N}{2} + \frac{L}{2a}\right)\ln\left(1 + \frac{L}{Na}\right) \right]$$

$$= k_B \left[ N\ln 2 - \left(\frac{N}{2} - \frac{L}{2a}\right)\ln\left(1 - \frac{L}{Na}\right) - \left(\frac{N}{2} + \frac{L}{2a}\right)\ln\left(1 + \frac{L}{Na}\right) \right] \qquad \text{——— ①}$$

$\because$ for $x \ll 1$

$$\ln(1+x) \simeq x - \frac{x^2}{2}$$

$$\Rightarrow \ln\left(1 - \frac{L}{Na}\right) \simeq \left(-\frac{L}{Na}\right) - \frac{L^2}{2N^2a^2}$$

$$\ln\left(1 + \frac{L}{Na}\right) \simeq \frac{L}{Na} - \frac{L^2}{2N^2a^2}$$

$\Rightarrow$ Thus, equation ① becomes.

$$S \simeq k_B \left[ N\ln 2 + \left(\frac{N}{2} - \frac{L}{2a}\right)\left(\frac{L}{Na} + \frac{L^2}{2N^2a^2}\right) - \left(\frac{N}{2} + \frac{L}{2a}\right)\left(\frac{L}{Na} - \frac{L^2}{2N^2a^2}\right) \right]$$

$$= k_B \left[ N\ln 2 + \cancel{\frac{NL}{2Na}} + \frac{NL^2}{4N^2a^2} - \frac{L^2}{2a^2N} - \cancel{\frac{L^3}{2^2a^3N^2}} - \cancel{\frac{NL}{2Na}} + \frac{NL^2}{4N^2a^2} - \frac{L^2}{2Na^2} + \cancel{\frac{L^3}{4N^2a^3}} \right]$$

$$= k_B \left[ N\ln 2 + \frac{1}{2}\frac{L^2}{Na^2} - \frac{L^2}{a^2N} \right] = k_B \left[ N\ln 2 + \frac{L^2}{Na^2}\left(\frac{1}{2} - 1\right) \right]$$

$$\Rightarrow \quad S \simeq k_B \left[ N \ln 2 - \frac{L^2}{2Na^2} \right]$$

The energy comes from the gravity potential energy.

$$E(L) = -mgL$$

(c) Temperature is defined as

$$\frac{1}{T} = \left( \frac{\partial S}{\partial E} \right)_{V,N}$$

$$= \left[ \frac{\partial S}{\partial(-mgL)} \right] = \left( \frac{\partial\, k_B \ln(\Gamma(L))}{\partial(-mgL)} \right)$$

$$= \frac{k_B}{-mg} \left( \frac{\partial \ln \Gamma(L)}{\partial L} \right)$$

$$= -\frac{k_B}{mg} \frac{\partial}{\partial L} \left( \frac{-L^2}{2Na^2} \right) = \frac{k_B L}{mgNa^2}$$

therefore,

$$\boxed{T = \frac{mgNa^2}{k_B L}}$$

(d) From ⓒ part, it is clear that

$L = \dfrac{mgNa^2}{k_B T}$ which shows that $L \propto mg$.

## QNO2

(a) For a two dimensional ideal gas of N particles, partition function is

$$Z = \frac{Z_1^N}{N!}$$ where; N! factor represent indistinguisble N particles.

$$\because \frac{Z_1^N}{N!} = \frac{1/h^{2N}}{N!}\int d^{2N}q \, d^{2N}p \, e^{-\beta \sum_{i=1}^{N} p_i^2/2m}$$

$$= \frac{A^N}{h^{2N}N!}\int d^{2N}p \, e^{-\beta \sum_{i=1}^{N} \frac{p_i^2}{2m}} .$$ $\because d^{2N}q = A^N$

$$= \frac{A^N}{N!\, h^{2N}}\int \prod_{i=1}^{N} d^2p_i \, e^{-\beta \sum_i \frac{p_i^2}{2m}}$$

$$\because \int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$= \frac{A^N}{N!\, h^{2N}}\left(\frac{1}{2}\sqrt{\frac{\pi}{\beta/2m}}\right)^{2N}$$

$$= \frac{A^N}{N!\, h^{2N}} \frac{1}{2^{2N}} \left(\sqrt{2m k_B T \pi}\right)^{2N}$$

$$Z = \frac{A^N}{N!}\left(\sqrt{\frac{2m k_B T \pi}{2^2 h^2}}\right)^{2N}$$

⇒ The Helmholtz free energy is F:

$$F = -k_B T \ln Z$$

$$= -k_B T \ln \left(\frac{A^N}{N!}\sqrt{\frac{2m k_B T \pi}{2^2 h^2}}\right)^{2N}$$

$$= -k_BT \ln \left[A\left(\frac{2mk_BT\pi}{4h^2}\right)\right]^N + k_BT\ln N!$$

$\Rightarrow$ Using Stirling approximation; for large N

$$\ln N! \simeq N\ln N - N$$

$$\Rightarrow F \simeq -k_BTN \ln\left(A\,\frac{mk_BT\pi}{2h^2}\right) + k_BTN\ln N - k_BTN$$

$$F = k_BTN\left[-1 - \ln\left(\frac{Amk_BT\pi}{2h^2N}\right)\right]$$

(b) Force exerted on one side of square wall by the gas. **F**

$\because$ Pressure $= \dfrac{\text{Force}}{\text{Area}} = \dfrac{F}{L}$ (for 2D, the P is force/length)

$$P = -\left(\frac{\partial F}{\partial V}\right)_T$$

$$P = -\left(\frac{\partial F}{\partial A}\right)_T \quad \text{for 2-D case.}$$

$$= k_BTN\frac{\partial}{\partial A}\ln A = \frac{k_BTN}{A} = \frac{k_BTN}{L^2}$$

$$\Rightarrow F = PL = \frac{k_BTN}{L^2}\cdot L$$

$$\Rightarrow \boxed{F = \frac{Nk_BT}{L}}$$

QN03

To evaluate mean force F b/w dipoles,

$$F = -\frac{\partial \langle V\rangle}{\partial R}$$

where

$$\langle V\rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}.$$

⇒ For the given classical system at temperature, the partition function is

$Z =$

$$Z = \int e^{-\beta H}\, d\Omega_1\, d\Omega_2$$

$$= \int e^{-\beta(K.E + P.E)}\, d\Omega_1\, d\Omega_2$$

∴ dipoles are fixed in space ⇒ K.E = 0

and P.E is

$$V = \phi = \left[\frac{3(\vec{M}_1\cdot\vec{R})(\vec{M}_2\cdot\vec{R}) - (\vec{M}_1\cdot\vec{M}_2)R^2}{R^5}\right] \quad ⑫$$

where

$$\vec{M}_1 = M_{1x}\hat{x} + M_{1y}\hat{y} + M_{1z}\hat{z}$$

$$= (M_1\sin\theta_1\cos\phi_1)\hat{x} + (M_1\sin\theta_1\sin\phi_1)\hat{y} + (M_1\cos\theta_1)\hat{z}$$

$$\vec{M}_2 = M_{2x}\hat{x} + M_{2y}\hat{y} + M_{2z}\hat{z}$$

$$= (M_2\sin\theta_2\cos\phi_2)\hat{x} + (M_2\sin\theta_2\sin\phi_2)\hat{y} + (M_2\cos\theta_2)\hat{z}.$$

$$\vec{R} = R\hat{z}.$$

$\Rightarrow$ P.E from Eq ⑫ becomes.

$$\phi = \frac{M_1 M_2}{R^3}\left[2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1-\phi_2)\right]$$

$$\because \; d\Omega_1 = \sin\theta_1 \, d\theta_1 \, d\phi_1$$
$$d\Omega_2 = \sin\theta_2 \, d\theta_2 \, d\phi_2 .$$

$\Rightarrow$ Therefore, partition function becomes

$$Z = \int e^{-\beta\phi} (\sin\theta_1 \, d\theta_1 d\phi_1)(\sin\theta_2 \, d\theta_2 \, d\phi_2)$$
$$= \int e^{-\frac{\beta M_1 M_2}{R^3}\left[2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1-\phi_2)\right]} . \sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi_1 \, d\phi_2$$

Let
$$\lambda = \frac{\beta M_1 M_2}{R^3}$$

$\Rightarrow$ Given that, $\lambda << 1$

$$\Rightarrow e^{-\lambda\left[2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1-\phi_2)\right]} \simeq 1 - \lambda A + \frac{\lambda^2 A^2}{2}$$

$$A = 2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1-\phi_2) .$$

$$\Rightarrow Z = \int e^{-\lambda A} \sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi_1 \, d\phi_2$$
$$= \int \left(1 - \lambda A + \frac{\lambda^2 A^2}{2}\right) \sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi_1 \, d\phi_2 .$$

$$Z = \int (1) \sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi_1 \, d\phi_2 - \lambda \int \Big(2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cdot$$

$$\cos(\phi_1 - \phi_2)\Big) \sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi_1 \, d\phi_2 + \frac{\lambda^2}{2} \int \left[2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)\right]^2 \cdot$$

$$\sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_1 \, d\phi_1 \, d\phi_2$$

$$= I_1 - \lambda I_2 + \frac{\lambda^2}{2} I_3$$

$\theta : 0 \to \pi$ , $\phi : 0 \to 2\pi$

$$I_1 = \int_0^{\pi} \sin\theta_1 \, d\theta_1 \int_0^{\pi} \sin\theta_2 \, d\theta_2 \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2$$

$$I_1 = 16\pi^2$$

And $I_2 = 2 \int \left\{ \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) \right\} \sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi_1 \, d\phi_2$

$$= 0$$

⇒ Thats why we cannot ignore $\lambda^2 A$ term while expanding $e^{-\lambda A}$.

$$Z = 16\pi^2 + \frac{\lambda^2}{2} I_3 = 16\pi^2 + \frac{\lambda^2}{2} Z'$$

$$I_3 = \int \left[2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)\right]^2 \sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi_1 \, d\phi_2$$

$$Z' = \int 4\cos^2\theta_1 \cos^2\theta_2 \sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \ (4\pi^2) + \int \sin^2\theta_1 \sin^2\theta_2 \cos^2(\phi_1-\phi_2) \cdot$$

$$\sin\theta_1 \sin\theta_2 \, d\theta_1 d\theta_2 \, d\phi_1 d\phi_2 - 4\int \cos\theta_1 \cos\theta_2 \sin^2\theta_1 \sin^2\theta_2 \cos(\phi_1-\phi_2) \, d\theta_1 d\theta_2 \cdot$$

$$d\phi_1 d\phi_2 .$$

$$= 2\left(\frac{32\pi^2}{9} + \frac{4\pi^2}{9}\right)$$

$$\Rightarrow Z = 16\pi^2 + \lambda^2\left(\frac{32}{9} + \frac{4}{9}\right)\pi^2$$

$$= 16\pi^2 + \lambda^2\left(\frac{36}{9}\right)\pi^2$$

$$= (16 + 4\lambda^2)\pi^2 \qquad \because \lambda = \beta\frac{M_1M_2}{R^3}$$

$$\because \langle v\rangle = -\frac{1}{Z}\frac{\partial Z}{\partial\beta}$$

$$\Rightarrow \frac{\partial Z}{\partial\beta} = \frac{\partial}{\partial\beta}\left(16\pi^2 + 4\pi^2\beta^2\frac{M_1^2M_2^2}{R^6}\right)$$

$$= \frac{4\pi^2}{R^6}M_1^2M_2^2 \, 2\beta = (8\pi^2M_1^2M_2^2)\beta R^{-6}$$

$$\Rightarrow \frac{1}{Z}\frac{\partial Z}{\partial\beta} = \frac{8\pi^2M_1^2M_2^2\beta R^{-6}}{(16\pi^2 + 4\beta^2M_1^2M_2^2R^{-6}\pi^2)}$$

$$= \frac{M_1^2M_2^2\beta R^{-6}}{\left(2 + \frac{\beta^2M_1^2M_2^2}{2}R^{-6}\right)} \qquad \because \text{let } M_1^2M_2^2\beta = C$$

$$\frac{M_1^2M_2^2\beta^2}{2} = D$$

$$= \frac{CR^{-6}}{(2 + DR^{-6})}$$

$$\Rightarrow \langle v\rangle = -\frac{CR^{-6}}{2 + DR^{-6}}$$

$\Rightarrow$ Mean force between the dipoles is,

$$F = -\frac{\partial \langle V \rangle}{\partial R}$$

$$= \frac{\partial}{\partial R}\left[ CR^{-6}\left(2+DR^{-6}\right)^{-1}\right]$$

$$= \frac{-6CR^{-7}}{\left(2+DR^{-6}\right)} + \frac{CR^{-6}}{\left(2+DR^{-6}\right)^{2}}(-1)(-6)DR^{-7}$$

$$= \frac{-6CR^{-7}}{\left(2+DR^{-6}\right)} + \frac{6CDR^{-13}}{\left(2+DR^{-6}\right)^{2}}$$

## QNO4

The ensemble average $\langle f \rangle$ of the function $f(q,P)$ is

$$\langle f \rangle = \frac{\int f(q,P)\, e^{-\beta H(q,P)}\, dw}{\int e^{-\beta H(q,P)}\, dw}$$

$$\because dw \equiv d^{3N}q\, d^{3N}P.$$

$\Rightarrow$ If $f = U - H(q,P)$ then

$$\langle f \rangle = \langle U - H(q,P) \rangle = \frac{\int [U - H(q,P)]\, e^{-\beta H(q,P)}\, d^{3N}q\, d^{3N}P}{\int e^{-\beta H(q,P)}\, d^{3N}q\, d^{3N}P}$$

$$= \frac{\int U e^{-\beta H(q,P)}\, d^{3N}q\, d^{3N}P}{\int e^{-\beta H(q,P)}\, d^{3N}q\, d^{3N}P} - \frac{\int H(q,P)\, e^{-\beta H(q,P)}\, d^{3N}q\, d^{3N}P}{\int e^{-\beta H(q,P)}\, d^{3N}q\, d^{3N}P}$$

$$\langle U - H(q,p)\rangle = \frac{U \cancel{\int e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p}}{\cancel{\int e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p}} - \frac{\int H(q,p)\, e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p}{\int e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p}$$

$$= U - \frac{\int H(q,p)\, e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p}{\int e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p}$$

$$\because U = \langle H\rangle = \langle E\rangle$$

$$= U - \langle H\rangle = 0$$

⇒ Now using the expression,

$$\int [U - H(q,p)]\, e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p = 0 \quad \text{Ⓐ}$$

we have to show that mean square fluctuation in the energy is

$$\langle (\Delta E)^2\rangle \equiv \langle E^2\rangle - \langle E\rangle^2 = -\left(\frac{\partial U}{\partial \beta}\right)$$

⇒ From Eq Ⓐ;

$$\int U e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p = \int H(q,p)\, e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p$$

$$U \int e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p = \int H(q,p)\, e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p$$

$$\Rightarrow U = \frac{\int H(q,p)\, e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p}{\int e^{-\beta H(q,p)} d^{3N}q\, d^{3N}p}$$

$$U = \left[\int e^{-\beta H(q,p)} H(q,p)\, d^{3N}q\, d^{3N}p\right]\left[\int e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p\right]^{-1}$$

$\Rightarrow$ differentiate w.r.t $\beta$,

$$\frac{\partial U}{\partial \beta} = \left[-\int H^2(q,p)\, e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p\right]\left[\int e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p\right]^{-1} +$$

$$(-1)\left[\int e^{-\beta H(q,p)} H(q,p)\, d^{3N}q\, d^{3N}p\right]\left[\int e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p\right]^{-2}$$

$$\left(-\int H(q,p)\, e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p\right)$$

$$= \frac{-\int H^2\, e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p}{\int e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p} + \frac{\left[\int H\, e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p\right]^2}{\left[\int e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p\right]^2}$$

$$= -\frac{\int H^2 e^{-\beta H}\, d^{3N}q\, d^{3N}p}{\int e^{-\beta H}\, d^{3N}q\, d^{3N}p} + \left[\frac{\int H e^{-\beta H}\, d^{3N}q\, d^{3N}p}{\int e^{-\beta H}\, d^{3N}q\, d^{3N}p}\right]^2 \quad \text{——} \; (X)$$

$$= -\langle H^2 \rangle + \langle H \rangle^2$$

$$\Rightarrow \quad -\left(\frac{\partial U}{\partial \beta}\right) = +\langle H^2 \rangle - \langle H \rangle^2$$

$$\because \; \langle E \rangle = \langle H \rangle$$

$$\Rightarrow \langle (\Delta E)^2 \rangle \equiv \langle E^2 \rangle - \langle E \rangle^2 = -\left(\frac{\partial U}{\partial \beta}\right) \quad \text{——} \; (\beta)$$

$$\because \; \frac{\partial}{\partial \beta} = \frac{\partial T}{\partial \beta} \cdot \frac{\partial}{\partial T} = -kT^2 \frac{\partial}{\partial T}$$

$$\Rightarrow \left(\frac{\partial U}{\partial \beta}\right) = -kT^2\left(\frac{\partial U}{\partial T}\right) = -kT^2 C_V \quad \text{—— ©}$$

$\Rightarrow$ Compare Ⓑ and ©;

$$\langle(\Delta E)^2\rangle \equiv \langle E^2\rangle - \langle E\rangle^2 = -\left(\frac{\partial U}{\partial \beta}\right) = kT^2\left(\frac{\partial U}{\partial T}\right) = kT^2 C_V$$

which is the required condition.

## QNO5

We have to show that for canonical ensemble, the mean cube fluctuation in the energy is

$$\langle(\Delta E)^3\rangle = k^2\left[T^4\left(\frac{\partial C_V}{\partial T}\right)_V + 2T^3 C_V\right] \quad \text{—— ①}$$

$\because \quad \Delta E = E - \langle E\rangle$

$\Rightarrow \quad (\Delta E)^3 = (E - \langle E\rangle)^3$

using identity for cube, $(a-b)^3 = a^3 - b^3 + 3ab^2 - 3a^2b$

$$\Rightarrow (\Delta E)^3 = E^3 - \langle E\rangle^3 + 3\langle E\rangle\langle E\rangle^2 - 3E^2\langle E\rangle$$

$$\begin{aligned}\langle(\Delta E)^3\rangle &= \langle E^3\rangle - \langle E\rangle^3 + 3\langle E\rangle\langle E\rangle^2 - 3\langle E^2\rangle\langle E\rangle \\ &= \langle E^3\rangle - \langle E\rangle^3 + 3\langle E\rangle\left[\langle E\rangle^2 - \langle E^2\rangle\right] \\ &= \langle E^3\rangle - \langle E\rangle^3 - 3\langle E\rangle\left[\langle E^2\rangle - \langle E\rangle^2\right] \\ &= \langle E^3\rangle - \langle E\rangle^3 - 3\langle E\rangle\,\langle(\Delta E)^2\rangle \quad \text{—— ②}\end{aligned}$$

$\Rightarrow$ By compare ① and ②, it is clear that we have to prove

$$k^2\left[T^4\left(\frac{\partial C_v}{\partial T}\right)_v + 2T^3 C_v\right] = \langle E^3\rangle - \langle E\rangle^3 - 3\langle E\rangle\langle(\Delta E)^2\rangle \quad ③$$

∵ In canonical ensemble, the average of a function is defined as

$$\langle f\rangle = \frac{\int f(q,p)\, e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p}{\int e^{-\beta H(q,p)}\, d^{3N}q\, d^{3N}p}$$

$$\Rightarrow \quad U = \langle E\rangle = \frac{\int E\, e^{-\beta E}\, d^{3N}q\, d^{3N}p}{\int e^{-\beta E}\, d^{3N}q\, d^{3N}p}$$

assume: $dw = d^{3}q\, d^{3N}p$

$$\Rightarrow \quad \langle E\rangle = \frac{\int E\, e^{-\beta E}\, dw}{\int e^{-\beta E}\, dw}$$

take its derivative w.r.t '$\beta$'

$$\frac{\partial U}{\partial \beta} = \frac{\partial \langle E\rangle}{\partial \beta} = \frac{\partial}{\partial \beta}\left[\left\{\int E e^{-\beta E}\, dw\right\}\left\{\int e^{-\beta E}\, dw\right\}^{-1}\right]$$

[Recall from solution of last question Eq Ⓧ]

$$\frac{\partial U}{\partial \beta} = -\frac{\int E^2 e^{-\beta E}\, dw}{\int e^{-\beta E}\, dw} + \left(\frac{\int E e^{-\beta E}\, dw}{\int e^{-\beta E}\, dw}\right)^2$$

⇒ Eq ③ includes cube terms ⇒ take again derivative of $\frac{\partial \langle E\rangle}{\partial \beta}$ w.r.t $\beta$.

$$\frac{\partial^2 U}{\partial \beta^2} = \frac{\partial}{\partial \beta}\left[\frac{-\int E^2 e^{-\beta E} dw}{\int e^{-\beta E} dw}\right] + \frac{\partial}{\partial \beta}\left[\frac{\int E e^{-\beta E} dw}{\int e^{-\beta E} dw}\right]^2$$

$$= I + I'$$

$$I = -\frac{\partial}{\partial \beta}\left[\left(\int E^2 e^{-\beta E} dw\right)\left(\int e^{-\beta E} dw\right)^{-1}\right]$$

$$= \frac{\int E^3 e^{-\beta E} dw}{\int e^{-\beta E} dw} - (-1)\frac{\int E^2 e^{-\beta E} dw}{\left(\int e^{-\beta E} dw\right)^2}\int e^{-\beta E}(-E)\, dw$$

$$= \frac{\int E^3 e^{-\beta E} dw}{\int e^{-\beta E} dw} - \frac{\int E^2 e^{-\beta E} dw}{\int e^{-\beta E} dw}\,\frac{\int E e^{-\beta E} dw}{\int e^{-\beta E} dw}$$

$$I = \langle E^3 \rangle - \langle E^2 \rangle \langle E \rangle$$

⇒ Now,

$$I' = \frac{\partial}{\partial \beta}\left[\left(\int E e^{-\beta E} dw\right)^2 \left(\int e^{-\beta E} dw\right)^{-2}\right]$$

$$= \frac{\frac{\partial}{\partial \beta}\left(\int E e^{-\beta E} dw\right)^2}{\left(\int e^{-\beta E} dw\right)^2} + \left(\int E e^{-\beta E} dw\right)^2 \frac{\partial}{\partial \beta}\left(\int e^{-\beta E} dw\right)^{-2}$$

$$= \frac{2\int E e^{-\beta E} dw \int (-E^2) e^{-\beta E} dw}{\left(\int e^{-\beta E} dw\right)^2} + 2\,\frac{\left(\int E e^{-\beta E} dw\right)^2 \left(\int E e^{-\beta E} dw\right)}{\left(\int e^{-\beta E} dw\right)^3}$$

$$= (-2)\,\frac{\int E e^{-\beta E}\,dw}{\int e^{-\beta E}\,dw}\,\frac{\int E^{2} e^{-\beta E}\,dw}{\int e^{-\beta E}\,dw} + 2\left(\frac{\int E e^{-\beta E}\,dw}{\int e^{-\beta E}\,dw}\right)^{2}\left(\frac{\int E e^{-\beta E}\,dw}{\int e^{-\beta E}\,dw}\right)$$

$$I' = -2\langle E\rangle\langle E^{2}\rangle + 2\langle E\rangle^{2}\langle E\rangle$$

$$\Rightarrow \frac{\partial^{2} U}{\partial \beta^{2}} = \langle E^{3}\rangle - \langle E^{2}\rangle\langle E\rangle - 2\langle E^{2}\rangle\langle E\rangle + 2\langle E\rangle^{3}$$

$$= \langle E^{3}\rangle - 3\langle E^{2}\rangle\langle E\rangle + 3\langle E\rangle^{3} - \langle E\rangle^{3}$$

$$= \langle E^{3}\rangle - \langle E\rangle^{3} - 3\langle E^{2}\rangle\langle E\rangle + 3\langle E\rangle^{2}\langle E\rangle$$

$$= \langle E^{3}\rangle - \langle E\rangle^{3} - 3\langle E\rangle\left[\langle E^{2}\rangle - \langle E\rangle^{2}\right]$$

$$\frac{\partial^{2} U}{\partial \beta^{2}} = \langle E^{3}\rangle - \langle E\rangle^{3} - 3\langle E\rangle\langle(\Delta E)^{2}\rangle$$ —— ④

⇒ Compare ② and ④,

$$\langle(\Delta E)^{3}\rangle = \frac{\partial^{2} U}{\partial \beta^{2}}$$

$$= \frac{\partial}{\partial \beta}\left(\frac{\partial U}{\partial \beta}\right) = \frac{\partial}{\partial \beta}\left(-kT^{2}C_{v}\right)$$

$$\because \left(\frac{\partial U}{\partial \beta}\right) = -kT^{2}C_{v}$$

$$= -kT^{2}\frac{\partial}{\partial T}\left[-kT^{2}C_{v}\right]$$

$$\because \frac{\partial}{\partial \beta} = \frac{\partial T}{\partial \beta}\cdot\frac{\partial}{\partial T} = -kT^{2}\frac{\partial}{\partial T}$$

$$\Rightarrow \left\langle (\Delta E)^3 \right\rangle = +k^2T^2\left[\frac{\partial}{\partial T}\left(T^2 C_v\right)\right]$$

$$= k^2T^2\left[2T C_v + T^2 \frac{\partial C_v}{\partial T}\right]$$

$$= k^2\left[2T^3 C_v + T^4 \left(\frac{\partial C_v}{\partial T}\right)_v\right]$$

$$\Rightarrow \left\langle (\Delta E)^3 \right\rangle = k^2\left[T^4 \left(\frac{\partial C_v}{\partial T}\right)_v + 2T^3 C_v\right]$$

Hence proved.

$\Rightarrow$ For ideal gas,

$$U = \langle E \rangle = \frac{3}{2} NkT$$

$$\because \left\langle (\Delta E)^2 \right\rangle = kT^2 C_v$$

$$= kT^2 \left(\frac{\partial U}{\partial T}\right)_v = kT^2 \frac{3}{2} Nk = \frac{3}{2} k^2 N T^2$$

$$\langle E \rangle^2 = \left(\frac{3}{2} NkT\right)^2 = \frac{9}{4} N^2 k^2 T^2$$

$$\left\langle \langle E \rangle^2 \right\rangle = \langle E \rangle^2 = \frac{9}{4} N^2 k^2 T^2$$

$$\Rightarrow \left\langle \left(\frac{\Delta E}{U}\right)^2 \right\rangle = \frac{\left\langle (\Delta E)^2 \right\rangle}{\langle U^2 \rangle} = \frac{\frac{3}{2} \cancel{k^2T^2N}}{\frac{9}{4} \cancel{k^2T^2}N^{\cancel{2}}} = \frac{2}{3N}$$

$$\Rightarrow \boxed{\left\langle \left(\frac{\Delta E}{U}\right)^2 \right\rangle = \frac{2}{3N}}$$ Hence proved.

Similarly,

$$\left\langle \left(\frac{\Delta E}{U}\right)^3 \right\rangle = \frac{\langle (\Delta E)^3 \rangle}{\langle U^3 \rangle} =$$

$$\because \langle (\Delta E)^3 \rangle = k^2 \left[ T^4 \left(\frac{\partial C_v}{\partial T}\right)_v + 2T^3 C_v \right]$$

$$\because C_v = \frac{3}{2} Nk$$ ideal gas

$$\Rightarrow \left(\frac{\partial C_v}{\partial T}\right)_v = 0$$

$$\Rightarrow \langle (\Delta E)^3 \rangle = k^2 \left(2T^3 \frac{3}{2} Nk\right) = 3k^3 T^3 N$$

$$\Rightarrow \langle \langle E \rangle^3 \rangle = \langle E \rangle^3 = \left(\frac{3}{2} NkT\right)^3 = \frac{3^3}{2^3} N^3 k^3 T^3$$

$$\Rightarrow \left\langle \left(\frac{\Delta E}{U}\right)^3 \right\rangle = \frac{3\cancel{k^3}\cancel{T^3}N}{\frac{3^3}{2^3} N^3 \cancel{k^3}\cancel{T^3}} = \frac{8}{9N^2}$$

$$\Rightarrow \boxed{\left\langle \left(\frac{\Delta E}{U}\right)^3 \right\rangle = \frac{8}{9N^2}}$$ Hence proved

# ASSIGNMENT # 3
## SOLUTIONS

## Q NO 1

For a magnetic material, the first Law of thermodynamics is d

$$dU = dQ + dW$$

$$dU = TdS + HdM.$$

∴ Heat capacity at constant H and M;

$$C_H = \left(\frac{\partial U}{\partial T}\right)_H = T\left(\frac{\partial S}{\partial T}\right)_H$$

$$C_M = \left(\frac{\partial U}{\partial T}\right)_M = T\left(\frac{\partial S}{\partial T}\right)_M$$

Assume $S = S(T, H)$

$$dS = \left(\frac{\partial S}{\partial T}\right)_H dT + \left(\frac{\partial S}{\partial H}\right)_T dH$$

$$\Rightarrow \left(\frac{\partial S}{\partial T}\right) = \left(\frac{\partial S}{\partial T}\right)_H + \left(\frac{\partial S}{\partial H}\right)_T \left(\frac{\partial H}{\partial T}\right)$$

$$\left(\frac{\partial S}{\partial T}\right)_M = \left(\frac{\partial S}{\partial T}\right)_H + \left(\frac{\partial S}{\partial H}\right)_T \left(\frac{\partial H}{\partial T}\right)_M$$

$$T\left(\frac{\partial S}{\partial T}\right)_M = T\left(\frac{\partial S}{\partial T}\right)_H + T\left(\frac{\partial S}{\partial H}\right)_T\left(\frac{\partial H}{\partial T}\right)_M$$

$$T\left(\frac{\partial S}{\partial T}\right)_H - T\left(\frac{\partial S}{\partial T}\right)_M = -T\left(\frac{\partial S}{\partial H}\right)_T\left(\frac{\partial H}{\partial T}\right)_M$$

∵ Maxwell's Relation

$$\left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H$$

$$\Rightarrow \quad C_H - C_M = -T\left(\frac{\partial S}{\partial H}\right)_T\left(\frac{\partial H}{\partial T}\right)_M$$

$$\boxed{C_H - C_M = -T\left(\frac{\partial M}{\partial T}\right)_H\left(\frac{\partial H}{\partial T}\right)_M} \qquad \text{———} \; ①$$

**(b)** For paramagnetic material, obeying Curie's law

$$M = \chi H$$

$$\chi = \frac{N\mu^2}{3kT} = \frac{C}{T} \qquad \because \; C = \frac{N\mu^3}{3k}$$

$$\Rightarrow \quad M = \frac{C}{T}H.$$

$$\left(\frac{\partial M}{\partial T}\right)_H = -\frac{CH}{T^2}, \quad \left(\frac{\partial H}{\partial T}\right)_M = \frac{M}{C}$$

plug $\left(\frac{\partial M}{\partial T}\right)_H$ and $\left(\frac{\partial H}{\partial T}\right)_M$ in ① Eq.

$$C_H - C_M = -T\left(-\frac{CH}{T^2}\right)\frac{M}{C}$$

$$= \frac{HM}{T} = \frac{H^2 C}{T^2} \qquad \because M = \frac{C}{T}H$$

$$\Rightarrow \boxed{C_H - C_M = \frac{CH^2}{T^2}}$$

## QNO2

$\mu = ?$ , $T = 293k$

$\chi = 1.80 \times 10^{-6}$ mks

$$\mu_B = 9.27 \times 10^{-24} \frac{am}{m^2} = JT^{-1}$$

$\because$

$$M = \chi H$$

$$\chi = \frac{N\mu^2}{3kT}$$

$$k_B = 1.38 \times 10^{-23}\ J/k.$$

$$k_B = 1.38 \times 10^{-23} \times 10^{7}\ erg/k$$

$$\Rightarrow \mu^2 = \frac{3kT\chi}{N}$$

$$\mu = \sqrt{\frac{3\,(1.38\times10^{-23+7})\,(273)\,K\left(\frac{erg}{K}\right)\left(1.08\times10^{-4}\,\frac{emu}{gOe}\right)}{\frac{6.023\times10^{23}}{32\ g/mole}}}$$

$$\mu = 2.54\times10^{-20}\ erg/Oe\text{-}mole$$

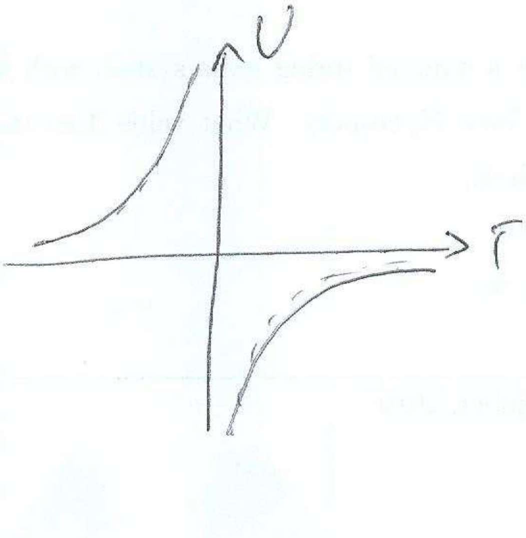
$$\boxed{\mu = 2.85\ \mu_B}$$

**QNO3**

(a) $\because\ T = \left(\frac{\partial U}{\partial S}\right)_V = -\frac{1}{2CU}$

$$\Rightarrow U = \frac{-1}{2CT}$$

(b) We assume $C>0$. The change of $U$ with $T$ is shown in the Figure.



# ASSIGNMENT # 4

# (SOLUTIONS)

## Q No 1

For the given gaseous system of magnetic atoms, the total energy is

$$E = K.E + P.E$$

$$= \frac{P^2}{2m} - \mu \cdot H = \frac{P^2}{2m} - \mu_B g J H S \qquad S = +1, -1$$

$$P.E_{\uparrow} = -\mu_{\uparrow} \cdot H = -\mu_B 2\left(\tfrac{1}{2}\right) H = -\mu_B H \qquad \because g = 2, \ J = \tfrac{1}{2}$$

$$P.E_{\downarrow} = -\mu_{\downarrow} \cdot H = \mu_B 2\left(\tfrac{1}{2}\right) H = \mu_B H$$

⇒ The canonical partition function for a single particle in the gas is,

$$Z_1 = \sum_{S=\pm1} \frac{1}{h^3} \int d^3q \, d^3P \, e^{-E_S \beta - E_{K.E}\beta}$$

$$= \sum_{S=\pm1} \frac{1}{h^3} \int d^3q \, d^3P \, e^{-\left(\frac{P^2}{2m} - S\mu_B H\right)\beta}$$

$$= \frac{V}{h^3} \int d^3P \, e^{\frac{-P^2\beta}{2m}} \cdot \left(e^{-\mu_B H \beta} + e^{+\mu_B H \beta}\right)$$

$$= \frac{V2}{h^3} \left(\frac{e^{\mu_B H\beta} + e^{-\mu_B H \beta}}{2}\right) \int d^3P \, e^{-P^2\beta/2m}$$

$$= \frac{2V}{h^3} \cosh \mu_B H \beta \left[\int dP \, e^{-\frac{\beta}{2m}P^2}\right]^3$$

$$\because \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\Rightarrow \int_0^\infty dP \, e^{-\frac{\beta P^2}{2m}} = \frac{1}{2}\sqrt{2\pi m k_B T}$$

$$\Rightarrow Z_1 = \frac{V}{h^3} \frac{(2\pi m k_B T)^{3/2}}{2^3} 2\cosh \mu_B H\beta .$$

$$= V \left(\frac{2\pi m k_B T}{2^2 h^2}\right)^{3/2} 2\cosh \mu_B H\beta .$$

$$= V \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} 2\cosh \mu_B H\beta . \text{ —— ①}$$

and the canonical partition function for N indistingusable non-interacting particles is

$$Z_N = \frac{1}{N!} Z_1^N$$

$$= \frac{1}{N!} \left[ V \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} 2\cosh \mu_B H\beta \right]^N$$

The relation b/w grand canonical partition function $Q_{GC}$ and canonical partition function is;

$$Q_{GC} = \sum_N \frac{1}{h^{3N}} \frac{1}{N!} \int d^{3N}q \, d^{3N}p \; e^{-\beta[H(q,p) - \mu N]}$$

$$= \sum_{N=0}^{\infty} Z(N) \, e^{\beta\mu N}$$

$$\Rightarrow Q_{GC} = \sum_{N=0}^{\infty} e^{\beta\mu N} Z(N) .$$

$$= \sum_{N=0}^{\infty} e^{\beta\mu N} \frac{1}{N!} Z_1^N$$

$$= \sum_{N=0}^{\infty} \frac{(Z_1 e^{\beta\mu})^N}{N!} = e^{(Z_1 e^{\beta\mu})}$$

is the grand partition function of a gaseous system of "magnetic atoms".

$$\left[ \because e^x = 1 + x + \frac{x^2}{2!} + \text{---} \right]$$

$$\Rightarrow Q_{GC} = e^{Z_1 e^{\beta\mu}} \quad \text{———} \quad ②$$

$\Rightarrow$ The magnetization is ;

$$M = -\left(\frac{\partial \Phi_{GC}}{\partial H}\right)_T$$

where $\Phi_{GC}$ is grand canonical potential, just like canonical potential $F$ = Helmholtz free energy in canonical ensemble.

$\Rightarrow$ Relation b/w $Q_{GC}$ and $\Phi_{GC}$ is;

$$Q_{GC} = e^{-\beta\Phi_{GC}}$$

$$\ln Q_{GC} = -\beta\Phi_{GC} = -\frac{1}{k_B T}\Phi_{GC}$$

$$\Rightarrow \Phi_{GC} = -k_B T \ln Q_{GC} \quad \text{———} \quad (a)$$

and $$M = k_B T\left(\frac{\partial}{\partial H}\ln Q_{GC}\right)_T$$

$$\because Q_{GC} = e^{Z_1 e^{\beta\mu}} \quad \text{———} \quad (b)$$

$$\Rightarrow \ln Q_{GC} = Z_1 e^{\beta\mu}$$

$$= \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} 2V e^{\beta\mu} \cosh \mu_B H\beta \quad \text{———} \quad ③$$

$$\Rightarrow M = k_B T \frac{\partial}{\partial H}\left[\left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} V e^{\beta\mu}\, 2\cosh \mu_B H\beta\right]_T$$

$$M = k_B T \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} V e^{\beta\mu} \frac{\partial}{\partial H}\left(2\cosh \mu_B H\beta\right)$$

$$= k_B T \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} V e^{\beta\mu} \frac{\partial}{\partial H}\left(e^{\mu_B H\beta} + e^{-\mu_B H\beta}\right)$$

$$= k_B T \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} V e^{\beta\mu} \mu_B \beta \left(e^{\mu_B H\beta} - e^{-\mu_B H\beta}\right)$$

$$= k_B \cancel{T} \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} V e^{\beta\mu} \frac{\mu_B}{k_B \cancel{T}} 2 \left(\frac{e^{\mu_B H\beta} - e^{-\mu_B H\beta}}{2}\right)$$

$$= \mu_B V 2\, e^{\beta\mu} \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} \sinh \mu_B H\beta$$

⇒ Thus, the magnetization of the given system is

$$M = 2\mu_B V e^{\beta\mu} \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} \sinh \mu_B H\beta$$

$\because$ $\langle N \rangle$ = Mean particle number $= -\frac{\partial}{\partial \alpha} \ln Q_{GC}$

where; $\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \mu}\frac{\partial \mu}{\partial \alpha} = \frac{\partial}{\partial \mu}\left(\frac{1}{\frac{\partial \alpha}{\partial \mu}}\right) = \frac{\partial}{\partial \mu}\left[\frac{1}{\frac{\partial}{\partial \mu}(-\beta\mu)}\right]$ $\because \alpha = -\beta\mu$

$$\Rightarrow \frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \mu}\left(\frac{-1}{\beta}\right) = -k_B T \frac{\partial}{\partial \mu}$$

$\langle N \rangle = k_B T \frac{\partial}{\partial \mu} \ln Q_{GC}$ $\quad\because$ From Eq ③

$$= k_B T\, 2V \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} \cosh \mu_B H\beta \, \frac{\partial}{\partial \mu} e^{\beta\mu}$$

$$= k_B T\, 2V \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} \cosh \mu_B H\beta \cdot \beta e^{\beta\mu}$$

$$\langle N\rangle = \cancel{k_B T}\left[2V\left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2}\cosh\mu_B H\beta \,.\, e^{\beta\mu}\right]\frac{1}{\cancel{k_B T}}$$

$$= V\left[2\cosh\mu_B H\beta \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} z\right]$$

where; $z$ = fugacity $= e^{\beta\mu}$

$$\langle N\rangle = V f(T)\, z \quad \text{—(c)}$$

where; $f(T)$ = function of temperature

$$f(T) = \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2}\left(2\cosh\mu_B H\beta\right) \quad \text{—(d)}$$

$$f(T) = f_{gas}(T) \times f_{spin}(T) \quad \text{—(e)}$$

⇒ To calculate heat flow $dQ$ as magnetic field reduces from $H \to 0$; at constant $T$ and $V$, we have to just find out the entropy change

$\because \quad dQ = TdS$

$\Delta Q = T\Delta S \qquad$ ; at constant T.

⇒ To compute the entropy, one way is to first obtain $F$ Helmholtz free energy as a function of $\langle N\rangle, V, T$

$\because \quad F = \phi_{GC} + \mu\langle N\rangle \qquad$ (For G.C ensemble)

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} \quad \text{—(f)}$$

$$\because F = \phi_{GC} + \mu\langle N\rangle \quad \text{—(4)}$$

where; $\phi_{GC} = -k_B T \ln Q_{GC} = -k_B T Z_1 e^{\beta\mu} = -k_B T Z_1 z$

⇒ plug $\Phi_{gc}$ in Eq ④;

$$F = -k_B T Z_1 z + \mu \langle N \rangle$$

$$= -k_B T Z_1 z + k_B T \ln z \langle N \rangle$$

$$= -k_B T z V \left(\frac{2\pi m k_B T}{4h^2}\right)^{3/2} 2\cosh \mu_B H \beta$$

$$+ k_B T \ln z \langle N \rangle$$

$\because \; z = e^{\beta\mu}$

$\Rightarrow \ln z = \beta\mu$

$\Rightarrow \mu = k_B T \ln z$.

$$= -k_B T z V f(T) + k_B T \ln z \langle N \rangle$$

$\because$ Using Eq (d)

⇒ From Eq (c); $\dfrac{\langle N \rangle}{V f(T)} = z \;\Rightarrow\; \ln\left(\dfrac{\langle N \rangle}{V f(T)}\right) = \ln z$.

$$\Rightarrow \quad F = -k_B T \left[z V f(T)\right] + k_B T \langle N \rangle \ln\left(\frac{\langle N \rangle}{V f(T)}\right)$$

$$= -k_B T \langle N \rangle + k_B T \langle N \rangle \ln\left(\frac{\langle N \rangle}{V f(T)}\right).$$

$$= k_B T \langle N \rangle \left[-1 + \ln \frac{\langle N \rangle}{V f(T)}\right]$$

$$\Rightarrow \quad S = -\left(\frac{\partial F}{\partial T}\right)_{V, \langle N \rangle}$$

$$= -\frac{\partial}{\partial T}\left[-k_B T \langle N \rangle + k_B T \ln \frac{\langle N \rangle}{V f(T)}\right]$$

$$= k_B \langle N \rangle - k_B \ln \frac{\langle N \rangle}{V f(T)} + k_B T \frac{\partial}{\partial T} \ln f(T)$$

$$= k_B \langle N \rangle \left[1 - \ln\left(\frac{\langle N \rangle}{V f(T)}\right) + T \frac{\partial}{\partial T} \ln f(T)\right]$$

$$S = k_B \langle N \rangle \left[ 1 + \ln\left( \frac{V f(T)}{\langle N \rangle} \right) + T \frac{\partial}{\partial T} \ln f(T) \right].$$

Since $f(T) = \left( \frac{2\pi m k_B T}{4 h^2} \right)^{3/2} \left( 2 \cos h \mu_B H \beta \right)$

$$= f_{gas}(T) \times f_{spin}(T)$$

where; $f_{spin}(T)$ has $H$ dependence only $\Rightarrow$

we are interested only $S_{spin}(T, H)$ to

evaluate $\Delta S_{spin} = S_f - S_i$

$$\Delta S = S_{spin}(T, H=0) - S_{spin}(T, H)$$

$$S_{spin} + S_{gas} = S$$

$$\Rightarrow S = k_B \langle N \rangle + k_B \langle N \rangle \ln\left( \frac{V f_{spin}(T)}{\langle N \rangle} \right) + k_B \langle N \rangle \ln\left( \frac{V f_{gas}(T)}{\langle N \rangle} \right)$$

$$+ \langle N \rangle k_B T \frac{\partial}{\partial T} \ln f_{spin}(T) + \langle N \rangle k_B T \frac{\partial}{\partial T} \ln f_{gas}(T).$$

where; $S_{spin}(T, H) = k_B \langle N \rangle \ln\left( \frac{V f_{spin}(T)}{\langle N \rangle} \right) + \langle N \rangle k_B T \frac{\partial}{\partial T} \ln f_{spin}(T)$

$$= k_B \langle N \rangle \ln\left( \frac{V}{\langle N \rangle} 2 \cos h \mu_B H \beta \right) + \langle N \rangle k_B T \frac{\partial}{\partial T} \ln f_{spin}(T)$$

where; $\frac{\partial}{\partial T} \ln f_{spin}(T) = \frac{1}{f_{spin}(T)} \frac{\partial}{\partial T} f_{spin}(T)$

$$= \frac{1}{2\cosh \mu_B H\beta} 2\sinh \mu_B H\beta \left(-\frac{\mu_B H}{k_B T^2}\right)$$

$$= \tanh \mu_B H\beta \left(-\frac{\mu_B H}{k_B T^2}\right)$$

Thus,

$$S_{spin}(T,H) = k_B \langle N\rangle \ln \frac{V}{\langle N\rangle} + k_B \langle N\rangle \ln 2\cosh \mu_B H\beta + \langle N\rangle k_B T\left(-\frac{\mu_B H}{k_B T^2}\right) \cdot \tanh \mu_B H\beta$$

$$\Rightarrow S_{spin}(T,H) = k_B \langle N\rangle \ln \frac{V}{\langle N\rangle} + k_B \langle N\rangle \ln 2\cosh \mu_B H\beta - \frac{\langle N\rangle \mu_B H}{T} \tanh \mu_B H\beta$$

$$= k_B \langle N\rangle \left[\ln \frac{V}{\langle N\rangle} + \ln 2\cosh \mu_B H\beta \quad \frac{\mu_B H}{k_B T} \tanh \mu_B H\beta\right]$$

$$= k_B \langle N\rangle \left[\ln \frac{V}{\langle N\rangle} + \ln 2\cosh \mu_B H\beta \quad \mu_B H\beta \tanh \mu_B H\beta\right]$$

$$\Rightarrow S_{spin}(T, H=0) = k_B \langle N\rangle \left[\ln \frac{V}{\langle N\rangle} + \ln 2 + 0\right]$$

$$\Rightarrow \Delta S = S_{spin}(T,H=0) - S_{spin}(T,H)$$

$$= k_B \langle N\rangle \left[\ln \frac{V}{\langle N\rangle} + \ln 2 - \ln \frac{V}{\langle N\rangle} - \ln 2 - \ln \cosh \mu_B H\beta + \mu_B H\beta \tanh \mu_B H\beta\right]$$

$$= k_B \langle N\rangle \left[\mu_B H\beta \tanh \mu_B H\beta - \ln \cosh \mu_B H\beta\right]$$

Thus; heat absorbed by the gas is then

$$\Delta Q = T\Delta S$$

$$= k_B T\langle N\rangle\left[\mu_B H\beta \tanh \mu_B H\beta - \ln \cosh \mu_B H\beta\right]$$

$$\Delta Q = k_B T\langle N\rangle\left[\frac{\mu_B H}{k_B T} \tanh \frac{\mu_B H}{k_B T} - \ln \cosh \frac{\mu_B H}{k_B T}\right]$$

which for $\frac{\mu_B H}{k_B T} \gg 1$ or $\mu_B H \gg k_B T$.

Can be simplify as;

$\Rightarrow$ For $y \gg 1$, $\tanh y \simeq 1$.    $y = \frac{\mu_B H}{k_B T}$

$$\Delta Q \simeq k_B T\langle N\rangle\left[\frac{\mu_B H}{k_B T} - \ln \cosh \frac{\mu_B H}{k_B T}\right]$$

$$= k_B T\langle N\rangle\left[y - \ln \cosh y\right] = k_B T\langle N\rangle\left[y - \ln\left(\frac{e^{y} + e^{-y}}{2}\right)\right]$$

$$= k_B T\langle N\rangle\left[y + \ln 2 - \ln\left(e^{y} + e^{-y}\right)\right]$$

$$= k_B T\langle N\rangle\left[y + \ln 2 - \ln e^{y}\left(1 + e^{-2y}\right)\right]$$

$$= k_B T\langle N\rangle\left[y + \ln 2 - \ln e^{y} - \ln\left(1 + e^{-2y}\right)\right]$$

$$= k_B T\langle N\rangle\left[\cancel{y} + \ln 2 \cancel{-y} - \ln\left(1 + e^{-2y}\right)\right]$$

$$= k_B T\langle N\rangle\left[\ln 2 - \ln(1 + 0)\right]$$

$\because$ $y \gg 1$

$\Rightarrow \frac{1}{e^{2y}} \ll 1$

$\simeq 0$

$$\Rightarrow \boxed{\Delta Q \simeq k_B T\langle N\rangle \ln 2}$$

## Q NO 2.)

(a) To find a relation b/w fugacity $z_g = e^{\beta\mu}$ and temperature T, pressure P; Starting from grand canonical partition function, then use a relation b/w P, V, T and $Q_{GC}$.

$$\frac{PV}{k_B T} = \ln Q_{GC}$$

to obtain a relation b/w $z_g$ and P, T.

⇒ The grand canonical partition function of an ideal classical gas is;

$$Q(T,V,\mu) = \sum_{N=0}^{\infty} e^{\beta\mu N} \frac{Z_{1c}^N}{N!}$$

where
$$Z_{c1} = \frac{V}{h^3}\int d^3P\, e^{-P^2/2m\beta}$$
$$= \frac{V}{h^3}\left[\int_0^{\infty} dP\, e^{-P^2\beta/2m}\right]^3$$

$$\because \int_0^{\infty} e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\Rightarrow Z_{c1} = \frac{V}{h^3}\frac{1}{2^3}\left(\frac{2\pi m}{\beta}\right)^{3/2} = V\left(\frac{2\pi m k_B T}{2^2 h^2}\right)^{3/2}$$

is the canonical partition function of one particle in the gas.

$$\Rightarrow Q_{GC}(T,V,\mu) = \sum_{N=0}^{\infty} \frac{\left(e^{\beta\mu} Z_{1C}\right)^N}{N!}$$

$$Q_{GC} = e^{Z_{1C} e^{\beta\mu}}$$

$$\because \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

$\Rightarrow$ According to the fundamental equation of thermodynamics; the grand potential $\phi$ is

$$\phi = F - \mu N = U - TS - \mu N \quad \text{——— (i)}$$

and $Q_{GC} = e^{-\beta\phi}$ (just like $Z_C = e^{-\beta F}$).

$\Rightarrow$ Using Euler's form of $U$;

$$U = TS - PV + \mu N$$

$$U - TS - \mu N = -PV \quad \text{——— (ii)}$$

$$\Rightarrow \phi = -PV$$

By compare (i) and (ii)

$$\Rightarrow Q_{GC} = e^{PV\beta} \quad \Rightarrow \ln Q_{GC} = PV\beta$$

$$\Rightarrow \frac{PV}{k_B T} = \ln Q_{GC}$$

$$= \ln e^{Z_{1C} e^{\beta\mu}}$$

$$\left(\frac{PV}{k_B T}\right) = Z_{1C} e^{\beta\mu}$$

$$\Rightarrow \frac{PV}{k_B T} = V\left(\frac{2\pi m k_B T}{2^2 h^2}\right)^{3/2} e^{\beta\mu}$$

$$\Rightarrow \frac{P}{(k_B T)^{1+3/2}} = \left(\frac{2\pi m}{2^2\hbar^2}\right)^{3/2} Z_g$$

$$\Rightarrow Z_g(T,P) = P(k_B T)^{-5/2}\left(\frac{\pi m}{2\hbar^2}\right)^{-3/2}$$

is the required relation b/w $Z_g$ and $(T,P)$

**(b)** For a box containing an ideal classical gas at pressure $P$ and temperature $T$, (there are $N_0$ absorbing sites on the walls); we have to evaluate the mean number of absorbed molecules $\langle N\rangle$.

⇒ Firstly; we will find average number of absorbed molecules per site and then multiply it with $N_0$ (total number of sites) to find $\langle N\rangle$.

⇒ For each absorbing site; grand canonical partition function given by

$$Q_{site} = \sum_{N=0}^{1} e^{\beta\mu N} e^{-\beta E N} = \sum_{N=0} e^{\beta\mu N} e^{+\beta E N}$$

$$= 1 + e^{\beta\mu} e^{\beta E} = 1 + Z_g e^{\beta E}$$

$$\Rightarrow Q_{site} = 1 + Z_g e^{\beta E}$$

⇒ The average number of absorbed molecules per site is

$$\langle N\rangle_{site} = -\frac{\partial}{\partial\alpha} \ln Q_{site}$$

$$\Rightarrow \langle N\rangle_{site} = -\frac{\partial}{\partial\alpha}\ln\left(1+Z_g e^{+\beta\varepsilon}\right)$$

$$\because \frac{\partial}{\partial\alpha} = \frac{\partial}{\partial Z_g}\frac{\partial Z_g}{\partial\alpha} = -e^{-\alpha}\frac{\partial}{\partial Z_g}$$

$$\because Z_g = e^{\beta\mu}$$

$$Z_g = e^{-\alpha}$$

$$\Rightarrow \frac{\partial}{\partial\alpha} = -Z_g\frac{\partial}{\partial Z_g}$$

$$\Rightarrow \langle N\rangle_{site} = Z_g\frac{\partial}{\partial Z_g}\ln\left(1+Z_g e^{\beta\varepsilon}\right)$$

$$\langle N\rangle_{site} = Z_g\frac{1}{\left(1+Z_g e^{\beta\varepsilon}\right)}e^{\beta\varepsilon} = \frac{1}{1+Z_g^{-1}e^{-\beta\varepsilon}}$$

⇒ So; for the $N_0$ sites, we have

$$\langle N\rangle = \frac{N_0}{1+Z_g^{-1}e^{-\beta\varepsilon}}$$

This average number of absorbed molecules can now be expressed in terms of the temperature and pressure of the gas by substituting for $Z_g = e^{\beta\mu}$ (fugacity) the function found in part (a).

$$\langle N\rangle = \frac{N_0}{1+P^{-1}(k_BT)^{5/2}\left(\frac{\pi m}{2h}\right)^{3/2}e^{-\beta\varepsilon}}$$

At a fixed temperature, this varies with pressure in a manner that might intuitively be expected. At high pressures, the density is high, molecules frequently approach the absorbing sites and $\langle N\rangle \simeq N_0$. At low P, the opposite is true and $\langle N\rangle \simeq 0$. The effect of T is also apparent. At high T, absorbed molecules are easily dislodged and a high P is required to keep the sites filled. At low T, a more modest pressure is sufficient

## Q No3

$$Z_{(1)} = \frac{1}{h^3}\int d^3P\, e^{-\frac{\beta P^2}{2m}} \int d^3r\, e^{-mgz\beta}$$

$$= \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \int dx \int dy \int_0^Z dz\, e^{-mgz\beta}$$

$$= \left(\frac{2\pi mkT}{h^2}\right)^{3/2} A \int_0^Z e^{-mgz\beta}\, dz.$$

$$= A\left(\frac{2\pi mkT}{h^2}\right)^{3/2} \left.\frac{e^{-mgz\beta}}{-mg\beta}\right|_0^Z$$

$$= \frac{A}{h^3}\left(2\pi mkT\right)^{3/2}\left(\frac{k_BT}{mg}\right)\left(1-e^{-mgZ\beta}\right)$$

$$= A\left(\frac{2\pi m}{h^2(mg)^{2/3}}\right)^{3/2}(kT)^{5/2}\left(1-e^{-mgZ\beta}\right)$$

$$\Rightarrow Z^N = A^N\left(\frac{2\pi m}{h^2(mg)^{2/3}}\right)^{\frac{3N}{2}}(kT)^{\frac{5N}{2}}\left(1-e^{-mgZ\beta}\right)^N$$

$$\because \mu = -k_B T \frac{\partial}{\partial N} \ln Z^N$$

$$\Rightarrow \mu = -k_B T \ln\left[A\left(\frac{2\pi m}{h^2 (mg)^{2/3}}\right)^{3/2} (kT)^{5/2} \left(1-e^{-mgz\beta}\right)\right]$$

$$= -k_B T \left[\ln\left(\frac{A}{mg}\right) + \ln\left(\frac{2\pi m}{h^2}\right)^{3/2} + \ln (kT)^{5/2} + \ln\left(1-e^{-mgz\beta}\right)\right].$$

$$\because P_0 = \frac{F}{A} = \frac{mg}{A}.$$

$$= -k_B T\left[-\ln P_0 + \ln\left(\frac{2\pi m}{h^2}\right)^{3/2} + \frac{5}{2}\ln(kT) + \ln\left(1-e^{-mgz\beta}\right)\right]$$

$$= -k_B T\left[-\ln P_0 + \ln\left(\frac{2\pi m}{h^2}\right)^{3/2} + \ln(kT) + \frac{3}{2}\ln(kT) + \ln\left(1-e^{-mgz\beta}\right)\right]$$

$$= -k_B T\left[-\ln P_0 + \ln\left(\frac{2\pi m}{h^2}\right)^{3/2} + \ln(kT) + \frac{3}{2}\ln(kT) + \ln\left(1-e^{-mgz\beta}\right)\right]$$

$$= -k_B T\left[-\ln\left(\frac{P_0}{P}P\right) + \ln\left(\frac{2\pi m}{h^2}\right)^{3/2} + \ln(kT) + \frac{3}{2}\ln T + \frac{3}{2}\ln k + \ln\left(1-e^{-mgz\beta}\right)\right]$$

$$= -k_B T\left[-\ln\left(\frac{P_0}{P}\right) + \ln\left(\frac{kT}{P}\right) + \frac{3}{2}\ln T + \sigma_0 + \ln\left(1-e^{-mgz\beta}\right)\right]$$

where $\sigma_0 = \ln\left(\frac{2\pi m k}{h^2}\right)^{3/2}$

$$\because \; P = P_0\, e^{-mgz\beta}$$

$$\Rightarrow \frac{P}{P_0} = e^{-mgz\beta} \quad \text{and} \quad \ln\left(\frac{P}{P_0}\right) = -mgz\beta$$

$$\mu = -k_B T\left[-mgz\beta + \ln\left(\frac{KT}{P}\right) + \frac{3}{2}\ln(T) + \ln\left(1-e^{-mgz\beta}\right) + \sigma_0\right]$$

$$\Rightarrow \mu = +mgz - k_B T\ln\left(\frac{k_B T}{P}\right) - \frac{3}{2}k_B T\ln(T) - k_B T\left(\ln\left(1-e^{-mgz\beta}\right) + \sigma_0\right).$$

## QNO4

**(a), (b)** To derive an expression for entropy S of an ideal quantum gas in terms of $\langle n_r \rangle$, the mean number of particles in state r; starting from the knowledge of partition functions $Q_{GC}$ for both quantum gases (Bose and Fermi)

$$\ln Q_{BE} = -\sum_r \ln\left(1-e^{-\beta\varepsilon_r-\alpha}\right) \text{ —— (i)}$$

$$\ln Q_{FD} = +\sum_r \ln\left(1+e^{-\beta\varepsilon_r-\alpha}\right) \text{ —— (ii)}$$

then find the grand potential $\phi$ for both systems

$$Q = e^{-\beta\phi}$$

$$\Rightarrow \ln Q = -\beta\phi$$

or

$$\left.\begin{aligned} \phi_{BE} &= -k_B T \ln Q_{BE} \\ \phi_{FD} &= -k_B T \ln Q_{FD}. \end{aligned}\right] \text{ —(iii)}$$

and differentiate it w.r.t T temperature at constant V, leads to required entropy.

$$S_{\substack{BE\\FD}} = -\left(\frac{\partial \phi_{\substack{B\\F}}}{\partial T}\right)_V \text{ —— (iv)}$$

⇒ Combined form of (i) and (ii) is;

$$\ln Q_{\substack{BE\\FD}} = \mp \sum_r \ln\left(1 \mp e^{-\beta E_r - \alpha}\right) \text{ ——(V)}$$

∵ Distribution function for Fermi and Bose systems are.

$$\langle n_r\rangle_{\substack{BE\\FD}} = \frac{1}{e^{\alpha+\beta E_r} \mp 1}$$

$$\Rightarrow \langle n_r\rangle_{\substack{B\\F}} \left(e^{\alpha+\beta E_r} \mp 1\right) = 1$$

$$\Rightarrow \langle n_r\rangle_{\substack{B\\F}} e^{\alpha+\beta E_r} = 1 \pm \langle n_r\rangle_{\substack{B\\F}}$$

$$\Rightarrow e^{\alpha+\beta E_r} = \frac{1 \pm \langle n_r\rangle_{\substack{B\\F}}}{\langle n_r\rangle_{\substack{B\\F}}}$$

or

$$\ln e^{\alpha+\beta E_r} = \ln\left(1 \pm \langle n_r\rangle_{\substack{B\\F}}\right) - \ln \langle n_r\rangle_{\substack{B\\F}}$$

$$\Rightarrow \quad \alpha + \beta\varepsilon_r = \ln\left(1 \pm \langle n_r\rangle_{\substack{B\\F}}\right) - \ln \langle n_r\rangle_{\substack{B\\F}}$$

$$\Rightarrow -\alpha - \beta\varepsilon_r = \ln\left(\frac{\langle n_r\rangle_{\substack{B\\F}}}{1 \pm \langle n_r\rangle_{\substack{B\\F}}}\right)$$

⇒ Now by plugging above relation in Eq (v)

$$\ln Q_{\substack{BE\\FD}} = \mp \sum_r \ln\left(1 \mp e^{-\alpha-\beta\varepsilon_r}\right)$$

$$= \mp \sum_r \ln\left(1 \mp \frac{\langle n_r\rangle_{\substack{B\\F}}}{1 \pm \langle n_r\rangle_{\substack{B\\F}}}\right)$$

$$= \mp \sum_r \ln\left(\frac{1 \pm \cancel{\langle n_r\rangle_{\substack{B\\F}}} \mp \cancel{\langle n_r\rangle_{\substack{B\\F}}}}{1 \pm \langle n_r\rangle_{\substack{B\\F}}}\right)$$

$$= \mp \sum_r \ln\left(\frac{1}{1 \pm \langle n_r\rangle_{\substack{B\\F}}}\right)$$

$$= \mp\left[\sum_r \ln 1 - \sum_r \ln\left(1 \pm \langle n_r\rangle_{\substack{B\\F}}\right)\right] \qquad \because \ln 1 = 0$$

$$\Rightarrow \quad \ln Q_{\substack{BE\\FD}} = \pm \sum_r \ln\left(1 \pm \langle n_r\rangle_{\substack{B\\F}}\right)$$

⇒ Using Eq (iii);

$$\phi_{\substack{BE\\FD}} = -k_B T \ln Q_{\substack{BE\\FD}}$$

$$= \mp k_B T \sum_r \ln\left(1 \pm \langle n_r\rangle_{\substack{B\\F}}\right)$$

differentiate w.r.t T at constant V,

$$S_{\substack{BE\\FD}} = -\left(\frac{\partial \phi_{\substack{B\\F}}}{\partial T}\right)_V = \pm k_B \frac{\partial}{\partial T}\left[T \sum_r \ln\left(1 \pm \langle n_r\rangle_{\substack{B\\F}}\right)\right]$$

$$\Rightarrow \quad S_{\substack{BE\\FD}} = \pm k_B \left[ \sum_r \ln\left(1 \pm \langle n_r \rangle_{\substack{B\\F}}\right) + T \frac{\partial}{\partial T} \ln\left(1 \pm \langle n_r \rangle_{\substack{B\\F}}\right) \right] \text{—— (vi)}$$

$$\because \quad \frac{\partial}{\partial T} = \frac{\partial}{\partial \beta}\frac{\partial \beta}{\partial T} = -\frac{1}{k_B T^2}\frac{\partial}{\partial \beta} \quad \text{—— (vii)}$$

$$\Rightarrow \quad \frac{\partial}{\partial \beta} \ln\left(1 \pm \langle n_r \rangle_{\substack{B\\F}}\right) = \frac{1}{1 \pm \langle n_r \rangle_{\substack{B\\F}}} \frac{\partial}{\partial \beta}\left(\pm \langle n_r \rangle_{\substack{B\\F}}\right) \text{—— (viii)}$$

$$\because \quad \langle n_r \rangle_{\substack{B\\F}} = \frac{1}{e^{\alpha+\beta\varepsilon_r} \mp 1}$$

$$\Rightarrow \quad \frac{\partial}{\partial \beta} \langle n_r \rangle_{\substack{B\\F}} = -\frac{1}{\left(e^{\alpha+\beta\varepsilon_r} \mp 1\right)^2} \frac{\partial}{\partial \beta} e^{\alpha+\beta\varepsilon_r}$$

$$\because \alpha = -\beta\mu$$

$$= -\frac{e^{\beta(\varepsilon_r-\mu)}(\varepsilon_r-\mu)}{\left(e^{\alpha+\beta\varepsilon_r} \mp 1\right)^2} \qquad \text{plug in (viii)}$$

$$\Rightarrow \quad \frac{\partial}{\partial \beta} \ln\left(1 \pm \langle n_r \rangle_{\substack{B\\F}}\right) = \mp \frac{e^{\beta(\varepsilon_r-\mu)}}{1 \pm \langle n_r \rangle_{\substack{B\\F}}} \cdot \frac{(\varepsilon_r-\mu)}{\left(e^{\alpha+\beta\varepsilon_r} \mp 1\right)^2}$$

$$= \mp \frac{e^{\alpha+\beta\varepsilon_r}}{1 \pm \langle n_r \rangle_{\substack{B\\F}}} \frac{1}{\beta} \frac{(\beta\varepsilon_r - \beta\mu)}{\left(e^{\alpha+\beta\varepsilon_r} \mp 1\right)^2}$$

$$= \mp \left(\frac{\cancel{1 \pm \langle n_r \rangle_{\substack{B\\F}}}}{\cancel{\langle n_r \rangle_{\substack{B\\F}}}}\right) \frac{1/\beta}{\cancel{1 \pm \langle n_r \rangle_{\substack{B\\F}}}} \ln\left(\frac{1 \pm \langle n_r \rangle_{\substack{B\\F}}}{\langle n_r \rangle_{\substack{B\\F}}}\right) \cdot \langle n_r \rangle_{\substack{BE\\FD}}^{\cancel{2}}$$

$$= \mp \frac{1}{\beta} \langle n_r \rangle_{\substack{B\\F}} \ln\left(\frac{1 \pm \langle n_r \rangle_{\substack{B\\F}}}{\langle n_r \rangle_{\substack{B\\F}}}\right) \qquad \text{plug in (vii) and (vi)}$$

$$\Rightarrow \frac{\partial}{\partial T} \ln\left(1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}\right) = \frac{-1}{k_B T^2} \frac{\partial}{\partial \beta} \ln\left(1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}\right)$$

$$= \pm \frac{1}{k_B T^2} (k_B T) \langle n_\lambda\rangle_{\substack{B\\F}} \ln\left(\frac{1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}}{\langle n_\lambda\rangle_{\substack{B\\F}}}\right)$$

$$= \pm \frac{1}{T} \langle n_\lambda\rangle_{\substack{B\\F}} \ln\left(\frac{1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}}{\langle n_\lambda\rangle_{\substack{B\\F}}}\right)$$

Thus; $S_{\substack{BE\\FD}}$ becomes

$$S_{\substack{BE\\FD}} = \pm k_B \left[ \sum_\lambda \ln\left(1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}\right) \pm \sum_\lambda \langle n_\lambda\rangle_{\substack{B\\F}} \ln\left(\frac{1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}}{\langle n_\lambda\rangle_{\substack{B\\F}}}\right)\right]$$

$$= k_B \sum_\lambda \left[ \pm \ln\left(1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}\right) + \langle n_\lambda\rangle_{\substack{B\\F}} \ln\left(1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}\right) - \langle n_\lambda\rangle_{\substack{B\\F}} \ln \langle n_\lambda\rangle_{\substack{B\\F}}\right]$$

$$= -k_B \sum_\lambda \left[ \mp \ln\left(1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}\right) - \langle n_\lambda\rangle_{\substack{B\\F}} \ln\left(1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}\right) + \langle n_\lambda\rangle_{\substack{B\\F}} \ln \langle n_\lambda\rangle_{\substack{B\\F}}\right]$$

$$S_{\substack{BE\\FD}} = -k_B \sum_\lambda \left[ \langle n_\lambda\rangle_{\substack{B\\F}} \ln \langle n_\lambda\rangle_{\substack{B\\F}} \mp \left(1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}\right) \ln\left(1 \pm \langle n_\lambda\rangle_{\substack{B\\F}}\right)\right]$$

Means to say

$$S_{BE} = -k_B \sum_\lambda \left[ \langle n_\lambda\rangle_B \ln \langle n_\lambda\rangle_B - \left(1 + \langle n_\lambda\rangle_B\right) \ln\left(1 + \langle n_\lambda\rangle_B\right)\right]$$

$$S_{FD} = -k_B \sum_\lambda \left[ \langle n_\lambda\rangle_F \ln \langle n_\lambda\rangle_F + \left(1 - \langle n_\lambda\rangle_F\right) \ln\left(1 - \langle n_\lambda\rangle_F\right)\right]$$

**(c)** In the classical limit, $\langle n_\lambda\rangle \ll 1$

$$S_{BE} = -k_B \sum_\lambda \left[ \langle n_\lambda\rangle_B \ln \langle n_\lambda\rangle_B - \ln\left(1 + \langle n_\lambda\rangle_B\right) - \langle n_\lambda\rangle_B \ln\left(1 + \langle n_\lambda\rangle_B\right)\right]$$

$$\Rightarrow S_{BE} = k_B \sum_r \left[ \ln(1+\langle n_r\rangle_B) - \langle n_r\rangle_B \ln \langle n_r\rangle_B + \langle n_r\rangle_B \ln(1+\langle n_r\rangle_B) \right]$$

$$= k_B \sum_r \left[ \ln(1+\langle n_r\rangle_B) + \langle n_r\rangle_B \ln\left( \frac{1+\langle n_r\rangle_B}{\langle n_r\rangle_B} \right) \right]$$

$$= k_B \sum_r \left[ \ln(1+\langle n_r\rangle_B) + \langle n_r\rangle_B \ln\left( \frac{1}{\langle n_r\rangle_B} + 1 \right) \right]$$

$$\because \langle n_r\rangle_B \ll 1 \Rightarrow \frac{1}{\langle n_r\rangle_B} \gg 1$$

$$\Rightarrow \frac{1}{\langle n_r\rangle_B} + 1 \simeq \frac{1}{\langle n_r\rangle_B}$$

and $1+\langle n_r\rangle_B \simeq 1 \Rightarrow \ln(1+\langle n_r\rangle_B) \simeq \ln 1 = 0$

$$\Rightarrow S_{BE} \simeq k_B \sum_r \left[ 0 + \langle n_r\rangle_B \ln\left( \frac{1}{\langle n_r\rangle_B} \right) \right]$$

$$\boxed{S_{BE} \simeq - k_B \sum_r \langle n_r\rangle_B \ln \langle n_r\rangle_B}$$ in classical limit.

Similarly, for Fermi gas.

$$S_{FD} = -k_B \sum_r \left[ \langle n_r\rangle_F \ln \langle n_r\rangle_F + \ln(1-\langle n_r\rangle_F) - \langle n_r\rangle_F \ln(1-\langle n_r\rangle_F) \right]$$

$$= k_B \sum_r \left[ -\langle n_r\rangle_F \ln \langle n_r\rangle_F + \langle n_r\rangle_F \ln(1-\langle n_r\rangle_F) - \ln(1-\langle n_r\rangle_F) \right]$$

$$= k_B \sum_r \left[ \langle n_r\rangle_F \ln\left( \frac{1-\langle n_r\rangle_F}{\langle n_r\rangle_F} \right) - \ln(1-\langle n_r\rangle_F) \right]$$

$$= k_B \sum_r \left[ \langle n_r\rangle_F \ln\left( \frac{1}{\langle n_r\rangle_F} - 1 \right) - \ln(1-\langle n_r\rangle_F) \right]$$

$\Rightarrow$ for $\langle n_\lambda \rangle_F \ll 1$

$$\Rightarrow \quad \ln\left(\frac{1}{\langle n_\lambda\rangle_F} - 1\right) \simeq \ln\left(\frac{1}{\langle n_\lambda\rangle_F}\right)$$

$$= -\ln\langle n_\lambda\rangle_F$$

and $\ln(1 - \langle n_\lambda\rangle_F) \simeq \ln 1 = 0$

$\Rightarrow$

$$\boxed{S_{FD} \simeq -k_B \sum_\lambda \langle n_\lambda\rangle_F \ln\langle n_\lambda\rangle_F}$$ in classical limit

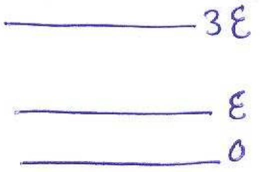
Thus, in classical limit $\langle n_\lambda\rangle \ll 1$

$$S_{BE} = S_{FD}$$

# ASSIGNMENT # 5
# (SOLUTIONS)

## QNO1

Given that, two particles (each of which) can be in any one of three quantum states. The system is in contact with a heat reservoir at temperature T.



(a) Z for Maxwell-Boltzmann statistics considering distinguisible particles;

$$Z = \sum_R e^{-\beta(\varepsilon_1 n_1 + \varepsilon_2 n_2 + ---)}$$

$$= \sum_{n_1, n_2, ---} e^{-\beta(\varepsilon_1 n_1 + \varepsilon_2 n_2 + ---)} \qquad \sum_i n_i = N = 2$$

$$\Rightarrow Z_{MB} = \left(e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} + e^{-\beta\varepsilon_3} + ---\right)^N \quad \because \text{derived in class}$$

$$\because \varepsilon_1 = 0, \quad \varepsilon_2 = \varepsilon, \quad \varepsilon_3 = 3\varepsilon, \quad N = 2$$

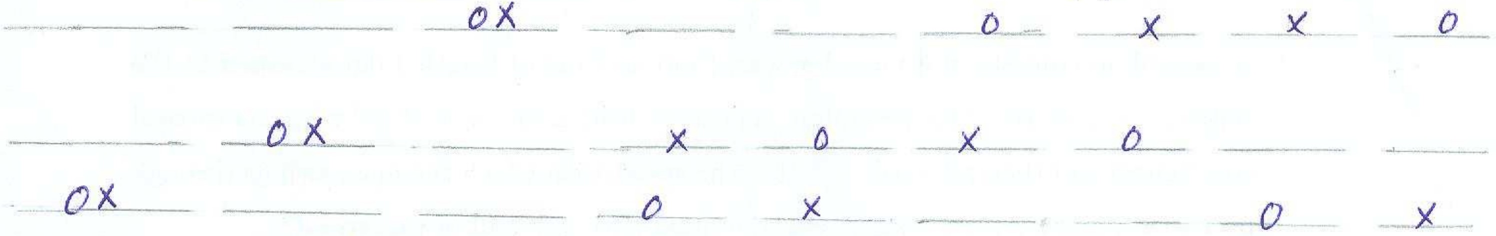
$$\Rightarrow Z_{MB} = \left(e^0 + e^{-\beta\varepsilon} + e^{-3\beta\varepsilon}\right)^2$$

$$= \left(1 + e^{-\beta\varepsilon} + e^{-3\beta\varepsilon}\right)\left(1 + e^{-\beta\varepsilon} + e^{-3\beta\varepsilon}\right).$$

$$= \left(1 + e^{-\beta\varepsilon} + e^{-3\beta\varepsilon} + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + e^{-4\beta\varepsilon} + e^{-3\beta\varepsilon} + e^{-4\beta\varepsilon} + e^{-6\beta\varepsilon}\right)$$

$$Z_{MB} = 1 + 2e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + 2e^{-3\beta\varepsilon} + 2e^{-4\beta\varepsilon} + e^{-6\beta\varepsilon}$$

## Alternate Method:- (Possible configurations)



$|200\rangle$ $|020\rangle$ $|002\rangle$ $|110\rangle$ $|110\rangle$ $|011\rangle$ $|011\rangle$ $|101\rangle$ $|101\rangle$

State: $|200\rangle$, $E = 2(0) = 0 = 1$

$|020\rangle$, $E = 2(\varepsilon) = 2\varepsilon$

$|002\rangle$, $E = 2(3\varepsilon) = 6\varepsilon$

$|110\rangle$, $E = 1(0) + 1(\varepsilon) = \varepsilon$

$|110\rangle$, $E = 1(0) + 1(\varepsilon) = \varepsilon$

$|011\rangle$, $E = 0(0) + 1(\varepsilon) + 1(3\varepsilon) = 4\varepsilon$

$|011\rangle$, $E = 0(0) + 1(\varepsilon) + 1(3\varepsilon) = 4\varepsilon.$

$|101\rangle$, $E = 1(0) + 1(3)\varepsilon = 3\varepsilon$

$|101\rangle$, $E = 3\varepsilon.$

$$\Rightarrow Z_{MB} = \sum_i e^{-\beta E_i}$$

$$= 1 + e^{-2\beta\varepsilon} + e^{-6\beta\varepsilon} + e^{-\beta\varepsilon} + e^{-\beta\varepsilon} + e^{-4\beta\varepsilon} + e^{-4\beta\varepsilon} + e^{-3\beta\varepsilon} + e^{-3\beta\varepsilon}$$

$$Z_{MB} = 1 + 2e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + 2e^{-3\beta\varepsilon} + 2e^{-4\beta\varepsilon} + e^{-6\beta\varepsilon}.$$

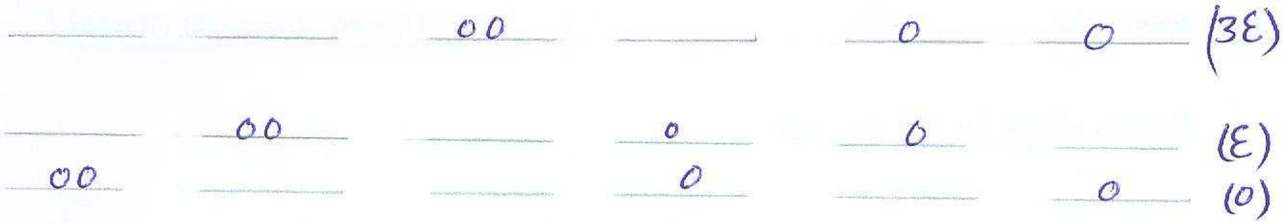
Both methods are consistent.

(b) For B.E Statistics, particles are indistinguisable.

$\Rightarrow$ Only 6 configurations are possible;

$$Z_{BE} = 1 + e^{-2\beta\varepsilon} + e^{-6\beta\varepsilon} + e^{-\beta\varepsilon} + e^{-4\beta\varepsilon} + e^{-3\beta\varepsilon}$$

$$\Rightarrow Z_{BE} = 1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + e^{-3\beta\varepsilon} + e^{-4\beta\varepsilon} + e^{-6\beta\varepsilon}$$



$$\begin{array}{cccccc} |2\,0\,0\rangle & |0\,2\,0\rangle & |0\,0\,2\rangle & |1\,1\,0\rangle & |0\,1\,1\rangle & |1\,0\,1\rangle \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ E_1 = 0 & E_2 = 2\varepsilon & E_3 = 6\varepsilon & E_4 = \varepsilon & E_5 = 4\varepsilon & E_6 = 3\varepsilon \end{array}$$

(c) $Z_{FD}$ for 2 particles obey Fermi-dirac statistics;

$$Z_{FD} = e^{-\beta\varepsilon} + e^{-4\beta\varepsilon} + e^{-3\beta\varepsilon}$$

$\because$ Only 3 possible configuration with $E_4 = \varepsilon$, $E_5 = 4\varepsilon$, $E_6 = 3\varepsilon$

**QNO 2**

Starting from the usual grand canonical expression for the particle number density in terms of temperature and chemical potential,

$$n = \int_0^\infty g(\varepsilon)\, f(\varepsilon)\, d\varepsilon \quad \text{——} \quad ①$$

where; $g(\varepsilon)$ = density of states

$f(\varepsilon)$ = Fermi dirac distribution function

$$= \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

$\because$ N = Number of particles = 2 (total number of states)

$\Rightarrow$ For 2-D Fermi gas,

$$N = 2\sum_k f(\vec{k}) = 2\left(\frac{L}{2\pi}\right)^2 \int_0^\infty f(\vec{k})\, d^2k \quad 2\left(\frac{L}{2\pi}\right)^2 \int_0^\infty f(\vec{k})\, 2\pi k\, dk = \frac{L^2}{\pi}\int_0^\infty k\, f(\vec{k})\, dk$$

$\because n = \text{number density} = \frac{N}{A}$

$$n = \frac{N}{A} = \frac{N}{L^2} = \frac{1}{\pi}\int_0^\infty k\, f(\vec{k})\, dk \text{ ——— } ②$$

$\because$

$$\varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}$$

$$k = \sqrt{\frac{2\varepsilon m}{\hbar^2}}$$

$$dk = \sqrt{\frac{2m}{\hbar^2}} \cdot \frac{1}{2} \cdot (\varepsilon)^{-1/2}\, d\varepsilon$$

$$k\,dk = \sqrt{\frac{2m}{\hbar^2}}\ \frac{1}{2}\sqrt{\frac{2\varepsilon m}{\hbar^2}}\ \varepsilon^{-1/2}\, d\varepsilon$$

$$= \left(\frac{2m}{\hbar^2}\right)\frac{1}{2}\, d\varepsilon$$

$\Rightarrow$ $k\,dk = \frac{m}{\hbar^2}\, d\varepsilon$ put in ②

$$\Rightarrow \quad n = \frac{1}{\pi}\int_0^\infty \frac{m}{\hbar^2}\, d\varepsilon = \int_0^\infty \frac{m}{\pi\hbar^2} f(\varepsilon)\, d\varepsilon \text{ ——— } ③$$

$\Rightarrow$ By compare ① and ③,

$$g(\varepsilon) = \frac{m}{\pi\hbar^2}$$

$$\Rightarrow \quad n = \int_0^\infty \frac{m}{\pi \hbar^2} \, f(\varepsilon) \, d\varepsilon$$

$$= \frac{m}{\pi \hbar^2} \int_0^\infty \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} \, d\varepsilon$$

$$\frac{\pi \hbar^2 n}{m} = \int_0^\infty \frac{e^{-\beta\varepsilon}}{\left(e^{-\beta\mu} + e^{-\beta\varepsilon}\right)} \, d\varepsilon$$

$$= \frac{-1}{\beta} \int_0^\infty \frac{-\beta e^{-\beta\varepsilon}}{\left(e^{-\beta\mu} + e^{-\beta\varepsilon}\right)} \, d\varepsilon$$

$$= -\frac{1}{\beta} \ln\left(e^{-\beta\mu} + e^{-\beta\varepsilon}\right) \Big|_0^\infty$$

$$= \frac{1}{\beta} \ln\left(e^{-\beta\mu} + e^{-\beta\varepsilon}\right) \Big|_\infty^0$$

$$= \frac{1}{\beta} \left[\ln\left(e^{-\beta\mu} + 1\right) - \ln\left(e^{-\beta\mu}\right)\right]$$

$$= \frac{1}{\beta} \ln\left(\frac{1 + e^{-\beta\mu}}{e^{-\beta\mu}}\right)$$

$$= \frac{1}{\beta} \ln\left(1 + e^{\beta\mu}\right)$$

$$\Rightarrow \quad n \frac{\pi \hbar^2}{m} = k_B T \ln\left(1 + e^{\beta\mu}\right) \quad \text{———} \; ④$$

At $T=0$ $\quad f(\varepsilon)=1$

$$\Rightarrow \quad n=\int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon$$

$$n=\frac{m}{\pi\hbar^2}\int_0^{\varepsilon_F} d\varepsilon$$

$$\Rightarrow \quad \varepsilon_F=\frac{n\pi\hbar^2}{m} \text{ ——— } ⑤$$

$$\varepsilon_F=\mu(T=0)=\frac{n\pi\hbar^2}{m}$$

By compare ④ and ⑤,

$$\varepsilon_F=k_BT\ln\left(1+e^{\beta\mu}\right)$$

$$\frac{\varepsilon_F}{k_BT}=\ln\left(1+e^{\beta\mu}\right)$$

$$\Rightarrow \quad e^{\beta\mu}=e^{\varepsilon_F/k_BT}-1$$

$$\Rightarrow \mu=k_BT\ln\left(e^{\varepsilon_F/k_BT}-1\right)$$

**QNO3**

∵ Average energy per particle in degenerate Fermi gas $=\langle\varepsilon\rangle=\frac{3}{5}E_F$

$\Rightarrow$ At $T=0$, $\varepsilon_F=\mu$ $\quad\Rightarrow \langle\varepsilon\rangle=\frac{3}{5}\mu$

We have to find the $\langle V_x^2 \rangle$ and $\langle V_x \rangle$

$$\because \quad \frac{1}{2} m V^2 = E$$

$$V^2 = \frac{2E}{m}$$

$$\Rightarrow \quad \langle V^2 \rangle = \frac{2}{m} \langle E \rangle$$

$$\because \; V^2 = V_x^2 + V_y^2 + V_z^2$$

if $V_x = V_y = V_z$

then $V^2 = 3 V_x^2$

$$\Rightarrow \left\langle 3 V_x^2 \right\rangle = \frac{2}{m} \langle E \rangle$$

$$\langle V_x^2 \rangle = \frac{2}{3m} \langle E \rangle$$

$$\because \; \langle E \rangle = \frac{3}{5} \mu$$

(per particle)

$$\langle V_x^2 \rangle = \frac{2}{3m} \cdot \frac{3}{5} \mu$$

$$\Rightarrow \boxed{\langle V_x^2 \rangle = \frac{2}{5} \frac{\mu}{m}}$$

$\Rightarrow$ Similarly, for $\langle V_x \rangle$

$$\sqrt{\frac{m}{2}} \, V = \sqrt{E}$$

$$\Rightarrow V = \sqrt{\frac{2}{m}} \sqrt{E}$$

$$\sqrt{3} \, V_x = \sqrt{\frac{2}{m}} \sqrt{E}$$

$$\Rightarrow \quad V_x = \sqrt{\frac{2}{3m}}\; E^{1/2}$$

$$\langle V_x \rangle = \sqrt{\frac{2}{3m}}\; \langle E^{1/2} \rangle \quad \text{———} \quad Ⓐ$$

Now calculate $\langle E^{1/2} \rangle$ for Fermi gas.

$$\because \quad \langle E^{1/2} \rangle = g \int_0^{\infty} D(E)\, f(E)\, E^{1/2}\, dE$$

where; $g$ = degeneracy factor

(Because of 2 electrons ↑↓ in each K state) $\rightarrow g = 2$

$$= 2 \int_0^{\infty} D(E)\, f(E)\, E^{1/2}\, dE$$

∵ At absolute zero temperature $\mu = E_F$ and $f(E) = 1$

$$\Rightarrow \quad \langle E^{1/2} \rangle = 2 \int_0^{E_F} D(E)\, E^{1/2}\, dE \quad \text{———} \quad ①$$

∵ $D(E)$ = density of states.

For 3-D Fermi gas; $D(E) = \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}$ ——— ②

plug $D(E)$ in Eq ①;

$$\Rightarrow \quad \langle E^{1/2} \rangle = 2 \int_0^{E_F} \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2 + 1/2}\, dE$$

$$= \frac{2V}{(2\pi)^2} \frac{(2m)^{3/2}}{\hbar^3} \int_0^{E_F} E\, dE$$

$$\Rightarrow \quad \langle E^{1/2}\rangle = \frac{2V}{(2\pi)^2}\frac{(2m)^{3/2}}{h^3}(2\pi)^3\int_0^{E_F} E\,dE$$

$$= 2\,\frac{(2\pi V)}{h^3}(2m)^{3/2}\,\frac{E_F^{\,2}}{2}$$

$$\langle E^{1/2}\rangle = \left(\frac{2\pi V}{h^3}\right)(2m)^{3/2}\,E_F^{\,2} \quad \text{———} \quad ③$$

⇒ Total number of electrons = N

$$N = g\int_0^{\infty} D(E)\,f(E)\,dE$$

$$= 2\int_0^{E_F} D(E)\,dE$$

at $T=0$

$f(E)=1$ and $\mu = E_F$

Recall from Eq ②; $D(E)_{3D} = \frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}E^{1/2}$

$$\Rightarrow \quad N = 2\int_0^{E_F}\frac{V}{(2\pi)^2}\frac{(2m)^{3/2}}{\hbar^3}E^{1/2}\,dE$$

$$= 2\,\frac{(2\pi V)}{h^3}(2m)^{3/2}\int_0^{E_F}E^{1/2}\,dE$$

$$= 2\left(\frac{2\pi V}{h^3}\right)(2m)^{3/2}\,\frac{E_F^{\,3/2}}{3/2}$$

$$= \left(\frac{2\pi V}{h^3}\right)(2m)^{3/2}\,E_F^{\,3/2} \quad \text{———} \quad ④$$

⇒ By comparing ③ and ④;

$$\frac{\langle E^{1/2}\rangle}{N} = \frac{\left(\frac{2\pi V}{h^3}\right)(2m)^{3/2} E_F{}^2}{\left(\frac{2\pi V}{h^3}\right)(2m)^{3/2}\frac{4}{3} E_F{}^{3/2}}$$

$$= \frac{3}{4} E_F^{(2-3/2)} = \frac{3}{4} E_F{}^{1/2} = \frac{3}{4}\mu^{1/2}$$

$$\Rightarrow \langle E^{1/2}\rangle = \frac{3}{4} N \mu^{1/2}$$ plug in Eq Ⓐ.

$$\Rightarrow \langle V_x\rangle = \sqrt{\frac{2}{3m}}\ \frac{3}{4} N \mu^{1/2}$$

$$= \sqrt{\frac{6\mu}{m4^2}}\ N$$

$$\Rightarrow \boxed{\langle V_x\rangle = \sqrt{\frac{3\mu}{m8}}\ N}$$ is required average of velocity in x-direction.

**Two Dimensional Fermi Gas**

**Answer:** 4

In two dimensional system with periodic boundary conditions we can solve the Schrodinger equation with momentum $k_x = \frac{2\pi}{L}n_x$ and $k_y = \frac{2\pi}{L}n_y$. The number of allowed values of $k$-space inside area ($L^2 = A$) is

$$\begin{aligned} \text{Number of allowed values of } k\text{-space} &= (\pi k_f^2)\left(\frac{L}{2\pi}\right)^2 \\ \Rightarrow \qquad \text{Total number of electrons} &= 2(\pi k_f^2)\left(\frac{L}{2\pi}\right)^2 \\ \Rightarrow \qquad n = \frac{N}{A} &= \frac{N}{L^2} = \frac{{k_f}^2}{2\pi} \end{aligned}$$

In two dimensions the density of particles in range $(k + dk)$ is $(2\pi k dk)$.

$$\begin{aligned} \Rightarrow \qquad n &= \int_0^{+\infty} g(\varepsilon)f(\varepsilon)d\varepsilon \\ N &= 2\sum_{\vec{k}} f(\vec{k}) \\ &= 2\left(\frac{L}{2\pi}\right)^2 \int_0^{\infty} f(\vec{k})d^2k \\ &= \frac{L^2}{2\pi^2}\int_0^{\infty} 2\pi k f(\vec{k})dk \\ &= \frac{L^2}{\pi}\int_0^{\infty} k f(\vec{k})dk \\ n &= \frac{N}{A} = \frac{N}{L^2} = \frac{1}{\pi}\int_0^{\infty} k f(\vec{k})dk \end{aligned}$$

Where $k$ can be calculated by using the relation of ground state energy,

$$\begin{aligned} \varepsilon(\vec{k}) &= \frac{\hbar^2 k^2}{2m} \\ \Rightarrow \qquad k &= \sqrt{\frac{2\varepsilon(\vec{k})m}{\hbar^2}} \\ \Rightarrow \qquad dk &= \sqrt{\frac{2m}{\hbar^2}} \cdot \frac{1}{2} \cdot \varepsilon(\vec{k})^{-1/2} d\varepsilon \\ \Rightarrow \qquad k dk &= \sqrt{\frac{2m}{\hbar^2}} \cdot \frac{1}{2} \cdot \sqrt{\frac{2\varepsilon(\vec{k})m}{\hbar^2}} \cdot \varepsilon(\vec{k})^{-1/2} d\varepsilon \\ &= \left(\frac{m}{\hbar^2}\right) d\varepsilon \end{aligned}$$

Thus electron density will become,

$$
\begin{aligned}
n &= \frac{1}{\pi}\int_0^\infty \left(\frac{m}{\hbar^2}\right) f(\varepsilon(\vec{k}))d\varepsilon \\
&= \int_0^\infty \left(\frac{m}{\pi\hbar^2}\right) f(\varepsilon(\vec{k}))d\varepsilon
\end{aligned}
$$

Where density of state $g(\varepsilon)$ is,

$$g(\varepsilon) = \frac{m}{\pi\hbar^2}$$

(b).

$$
\begin{aligned}
\because \quad E_f &= \frac{\hbar^2 {k_f}^2}{2m} \\
&= \frac{\hbar^2}{2m}(2\pi n) \qquad \because {k_f}^2 = 2\pi n \\
&= \frac{\pi n \hbar^2}{m} \\
&= \frac{3.14 \times 6 \times 10^{14} \times (1.055 \times 10^{-34})^2}{9.1 \times 10^{-31}} \\
&= 2.301 \times 10^{-19}\ J \\
&= 1.436\ eV
\end{aligned}
$$

## QNo5

(a) Using the relation for thermal de Broglie wavelength;

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

at room temperature, T=300k, for electron

$$\lambda = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 3.14 \times 9.11 \times 10^{-31} \times 1.38 \times 10^{-23} \times 300}}$$

$$\lambda = 4.3 \times 10^{-9}\ m$$

⇒ The requirement for the very applicability of the classical approximation is that the de-Broglie wavelength (its magnitude) λ be smaller than the interparticle separation, i.e.,

$$\lambda \ll \frac{L}{N^{1/3}}$$

$$\Rightarrow \quad \lambda^3 \ll \frac{L^3}{N} = \frac{V}{N} = \frac{1}{n}$$

$$\Rightarrow \quad \lambda^3 \ll \frac{1}{n}$$

$\because n = 8.5\times10^{28}\,m^{-3} \quad \Rightarrow \quad \frac{1}{n} = 1.176\times10^{-29}\,m^3$

and $\lambda^3 = (4.3\times10^{-9}\,m)^3 = 7.95\times10^{-26}\,m^3$

$\Rightarrow$ it is clear that

$$7.95\times10^{-26}\,m^3 \gg 1.17\times10^{-29}\,m^3$$

means to say that $\lambda^3 \not\ll \frac{1}{n}$

Thus, Boltzmann statistics cannot be applied to the mobile electrons in copper.

(b) Using the relation b/w number density and Fermi energy $E_F$;

$$E_F = \frac{h^2}{8m}\left(\frac{3n}{\pi}\right)^{2/3}$$

$$= \frac{(6.625\times10^{-34})^2\,(3\times8.5\times10^{28})^{2/3}}{8\times9.11\times10^{-31}\times(3.14)^{2/3}}$$

$$E_F = 1.1\times10^{-18}\ \text{J}$$

$\because$ $E = k_B T$ at Room temperature $= 300$k

$= (1.38\times10^{-23}\times300)$

$E = 4.14\times10^{-21}$ J

$\Rightarrow$ The condition for strongly degenerate gas

$$E_F \gg k_B T$$

$$1.1\times10^{-18}\ \text{J} \gg 4.14\times10^{-21}\ \text{J}$$

is fulfilled. Thus, room temperature sufficient to treat this system as degenerate electron gas.

# ASSIGNMENT # 6 (SOLUTION)

Q#1 (a) To find a relation b/w electron density 'n' and Fermi momentum; just find the total number of electrons in degenerate electron gas.

$$\because \quad N = g \sum_{\vec{k}} f(\varepsilon_{\vec{k}})$$

where $g$ = degeneracy factor $= 2$ and $f(\varepsilon_{\vec{k}})$ is Fermi dirac distribution function; for degenerate gas $f(\varepsilon) \rightarrow 1$.

and sum over all $\vec{k}$ states is

$$\sum_{\vec{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3k$$

$$\Rightarrow \quad N = 2\frac{V}{(2\pi)^3} \int 4\pi k^2 \, dk.$$

$\because P = \hbar k$

$\Rightarrow \frac{1}{\hbar} dP = dk$

$$= \frac{2(4\pi)V}{(2\pi)^3} \frac{1}{\hbar^3} \int_0^{P_F} P^2 dP$$

$$= \frac{8\pi V}{(2\pi)^3} \frac{1}{(h/2\pi)^3} \int_0^{P_F} p^2 dp.$$

$$\Rightarrow \quad N = \frac{8\pi V}{h^3} \frac{p_F^3}{3}$$

$$\Rightarrow \quad n = \frac{N}{V} = \frac{8\pi}{h^3} \frac{P_F^3}{3} \quad \text{or} \quad n = \frac{8\pi}{3}\left(\frac{P_F}{h}\right)^3 \text{ —— ①}$$

$\Rightarrow$ According to the given condition, we have to find $n$ for which $P_F = \frac{m_e c}{10}$, plug in ①

$$\Rightarrow \quad n = \frac{8\pi}{3}\left(\frac{m_e c}{10h}\right)^3$$

$$= \frac{8\times 3.14}{3}\left(\frac{9.11\times10^{-31}\times 3\times10^{8}}{10\times 6.625\times10^{-34}}\right)^3$$

$$\boxed{n = 5.8\times10^{32}\ m^{-3}}$$

(b) For a strong degenerate Fermi gas (under the approximation of zero valence), we get.

$$\langle E\rangle = U = \frac{3}{5} N \varepsilon_F$$

and $P = \frac{2}{3}\frac{U}{V}$

$$= \frac{2}{3}\frac{1}{V}\frac{3}{5}N\varepsilon_F = \frac{2}{5} n \varepsilon_F \qquad \because \varepsilon_F = \frac{p_F^2}{2m_e}$$

$$= \frac{2}{5} n \frac{p_F^2}{2m_e} = \frac{n p_F^2}{5 m_e}$$

$$= \frac{n}{5m_e}\left(\frac{3n}{8\pi}\right)^{2/3} h^2 \qquad \because p_F^{\,2} = \left(\frac{3n}{8\pi}\right)^{2/3} h^2$$

$$= \frac{(5.8\times10^{32})}{5\times 9.11\times10^{-31}}\left(\frac{3\times5.8\times10^{32}}{8\times3.14}\right)^{0.66}\left(6.625\times10^{-34}\right)^2$$

$$\boxed{P = 5.44\times10^{16}\ N/m^2}.$$

Q # 2

(a) We have to find a relation for average K.E U of the electrons and the density $n_e$, for the extremely relativistic electrons;
↓ (ignore rest energy)

$$\because \quad U = \langle E \rangle = \sum_{\vec{k}} f(\vec{k})\, \varepsilon_{\vec{k}}$$

where, $\sum_{\vec{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3k$.

$$= \frac{V}{(2\pi)^3} 4\pi \int k^2 dk$$

Use the approximation of strong degenerate electron gas and $\varepsilon \simeq PC$;

$$U = \frac{V}{(2\pi)^3}\, g\, 4\pi \int \varepsilon\, k^2 dk$$

$$= \frac{V}{(2\pi)^3}\, 2(4\pi) \int PC \left(\frac{\hbar^2 k^2}{\hbar^2}\right) \frac{1}{\hbar}\, dP$$

$$= \frac{8\pi V}{(2\pi)^3} \frac{C}{\hbar^3} \int_0^{P_F} p\, p^2 dp$$

$$= \frac{8\pi VC}{h^3} \int_0^{P_F} p^3 dp = \frac{8\pi VC}{h^3} \frac{P_F^4}{4}$$

$$\Rightarrow \boxed{U = \frac{2\pi VC}{h^3} P_F^4} \quad \text{— (A)}$$

$\because \varepsilon = PC$
$= \hbar kC$
$\Rightarrow dk = \frac{1}{\hbar} dP$.

$\because g =$ degeneracy $= 2$

$$\because \quad N = \sum_{\vec{k}} f(\varepsilon_{\vec{k}}).$$

where, sum over all $k$-states

$$\sum_{\vec{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3k$$

and $f(\varepsilon_{\vec{k}}) \rightarrow 1$ (for degenerate gas $(T=0)$)

$$\Rightarrow N = \frac{V}{(2\pi)^3} g \int d^3k$$

$$= \frac{V}{(2\pi)^3} 2 (4\pi) \int k^2 \, dk$$

$$= \frac{8\pi V}{(2\pi)^3} \frac{1}{\hbar^3} \int_0^{p_F} \hbar^2 k^2 \, d(\hbar k)$$

$\because p = \hbar k.$

$$= \frac{8\pi V}{(2\pi)^3} \left(\frac{2\pi}{h}\right)^3 \int_0^{p_F} p^2 \, dp$$

$$N = \frac{8\pi V}{h^3} \frac{p_F^3}{3}$$

$$\Rightarrow \boxed{\frac{N}{V} = n_e = \frac{8\pi}{3h^3} p_F^3} \quad \text{——} \quad Ⓑ$$

⇒ Now compare Ⓐ and Ⓑ,

$$U = \langle E_e \rangle = \left(\frac{2\pi C}{h^3} p_F^4\right) V$$

$$= \left(\frac{2\pi C}{h^3} p_F^4\right) \frac{V}{N} \cdot N = \left(\frac{2\pi C}{h^3} p_F^4\right) \frac{N}{n}$$

$$= \left(\frac{4(2\pi)}{3h^3} \cdot \frac{3}{4} p_F^3\right) p_F C \frac{N}{n}$$

$$= \left(\frac{8\pi}{3h^3} p_F^3\right) \frac{3}{4} p_F C \frac{N}{n}$$

$$= \cancel{n} \frac{3}{4} p_F C \frac{N}{\cancel{n}} = \frac{3}{4} C N p_F$$

$$\langle E \rangle = \frac{3}{4} C N \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3} = \frac{3}{4} hc \left(\frac{3n_e}{8\pi}\right)^{1/3} \cdot N$$

$$\Rightarrow \frac{\langle E_e \rangle}{N} = \frac{3}{4} hc \left(\frac{3n_e}{8\pi}\right)^{1/3}$$ is the required relation.

$\langle E_e \rangle$ per particle $= \frac{3}{4} hc \left(\frac{3n_e}{8\pi}\right)^{1/3}$.

$\Rightarrow$ Given that; $\rho = 10^9 \, g\, cm^{-3}$ and $n_e = 14 n_i$ (electron number density).

$$\because M = N m_e + 2N m_n$$

$N m_e =$ (Number of electrons) (mass of electron)

$2N m_n =$ (Number of nucleons) (mass of nucleon)

$$\Rightarrow M = N m_e + 2N m_n \qquad \because m_e \ll m_n$$
$$\simeq 2N m_n$$

$$\Rightarrow N = \frac{M}{2m_n} \quad \text{and} \quad n = \frac{N}{V} = \frac{M/2m_n}{M/\rho}$$

$$\Rightarrow n = \frac{\rho}{2m_n} \; ; \; n_i = \frac{\rho}{2m_i}$$

$$m_i = \frac{14g}{N_A} = \frac{14g}{6.023\times10^{23}} = 2.32\times10^{-23} g$$

$$\Rightarrow n_i = \frac{10^9 \, g\, cm^{-3}}{2\times 2.32\times10^{-23} g} = 2.155\times10^{31} \, cm^{-3}$$

$$\Rightarrow n_e = 14\,(2.155\times10^{31})$$
$$= 3.01\times10^{32} \, cm^{-3}$$
$$= 3.01\times10^{38} \, m^{-3}$$

$$\Rightarrow \langle E_e \rangle = \frac{3}{4} hc \left(\frac{3n_e}{8\pi}\right)^{1/3}$$
$$= \frac{3}{4} \times 6.625\times10^{-34} \times 3\times10^{8} \left(\frac{3\times 3.01\times10^{38}}{8\times3.14}\right)^{1/3}$$
$$\langle E_e \rangle = 5\times10^{-13} J$$

and Assume "ion gas" as "Maxwell-Boltzmann gas";

$$\langle E_i \rangle = \frac{3}{2} k_B T = \frac{3}{2}(1.38\times10^{-23})(10^8 K) = 2\times10^{-15} J$$

Obviously; $\boxed{\langle E_i \rangle \ll \langle E_e \rangle}$

**Q#3**

For a gas of N photons with number conservation;

$$N = 2V \int_0^\infty \frac{d^3k}{(2\pi)^3} \left(\frac{1}{e^{\beta\hbar c k - \beta\mu} - 1}\right).$$

$\Rightarrow$ 2 comes from the two polarizations of photons.

$$n = \frac{N}{V} = \frac{2}{(2\pi)^3}\, 4\pi \int_0^\infty \frac{k^2\, dk}{z^{-1} e^{\beta\hbar c k} - 1}$$

$\because z = e^{\beta\mu}$

$z =$ fugacity

$$= \frac{1}{\pi^2} \int \frac{k^2\, dk}{z^{-1} e^{\beta\hbar c k} - 1}$$

$$= \frac{1}{\pi^2 (\beta\hbar c)^3} \int_0^\infty \frac{(\beta\hbar c k)^2\, d(\beta\hbar c k)}{z^{-1} e^{\beta\hbar c k} - 1}$$

$(\because x = \beta\hbar c k)$
$$= \frac{1}{\pi^2} \frac{1}{(\beta\hbar c)^3} \int_0^X \frac{x^2\, dx}{z^{-1} e^{x} - 1}$$

Let
$$\int_0^\infty \frac{x^2\, dx}{z^{-1} e^{x} - 1} = \int_0^\infty \frac{z\, x^2 e^{-x}}{(1 - z e^{-x})}\, dx$$

where
$$\frac{z\, e^{-x}}{(1 - z e^{-x})} = z e^{-x} (1 - z e^{-x})^{-1}$$
$$= z e^{-x} \left[ 1 + z e^{-x} + z^2 e^{-2x} + \cdots\cdots \right]$$
$$= \sum_{p=1}^{\infty} z^p e^{-px}$$

$$\Rightarrow \int_0^\infty \frac{z x^2 e^{-x}}{(1 - z e^{-x})}\, dx = \sum_{p=1}^{\infty} z^p \int_0^\infty x^2 e^{-px}\, dx$$

$\because y = xp$

$$= \sum_{p=1}^{\infty} \frac{z^p}{p^3} \int_0^\infty e^{-y} y^2\, dy$$

$$\because \int_0^\infty x^n e^{-x/a} dx = a^{n+1} n!$$

or

$$\Rightarrow \int_0^\infty x^{z-1} e^{-x} dx = \Gamma(z) = (z-1)!$$

Therefore,

$$\int_0^\infty \frac{z x^2 e^{-x}}{(1-ze^{-x})} dx = \sum_{p=1}^{\infty} \frac{z^p}{p^3} \Gamma(3) = \sum_{p=1}^{\infty} \frac{z^p}{p^3} 2.$$

$$\Rightarrow n = \frac{2}{\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 \sum_{p=1}^{\infty} \frac{z^p}{p^3} \qquad \because g_3(z) = \sum_{p=1}^{\infty} \frac{z^p}{p^3}$$

$$= \frac{2}{\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 g_3(z)$$

$\Rightarrow$ The function $g_3(z)$ is finite at $z=1$, thus $n$ has a upper limit. There is condensation to the zero momentum state when $n$ exceeds the critical density.

$$n_c = \frac{2}{\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 g_3(1)$$

$\Rightarrow$ At constant density $n$, the condensation occurs below the critical temperature.

$$\left(\frac{\pi^2 n}{2}\right) \frac{\hbar^3 c^3}{k_B^3} \frac{1}{g_3(1)} = T_c^3$$

$$\Rightarrow T_c = \frac{\hbar c}{k_B} \left(\frac{\pi^2 n}{2 g_3(1)}\right)^{1/3}.$$

Q No 5

(a) The universe is pervaded by $T = 3k$ black body radiation, this radiation arose from the adiabatic exponsion of photon cloud.

⇒ The energy density of black body radiation is

$$u = aT^4 \quad \text{——} ①$$

so the total energy is

$$E \sim VT^4 \quad \text{——} ②$$

Using first law of thermodynamics,

$$dU = dQ - PdV$$
$$dU = TdS - PdV$$

$$\therefore C_V = \left(\frac{\partial U}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \propto VT^3$$

> For a reversible adiabatic expansion, $\Delta S = 0$ means entropy remains constant.

$$\therefore \Delta S = \int_0^T \frac{C_V}{T} dT$$

$$\Delta S = S(T=T) - S(T=0)$$

Using third law of thermodynamics $S(T=0) = 0$

$$\Rightarrow \quad S(T) = \int_0^T \frac{C_V}{T}\, dT$$

$$\sim V \int_0^T \frac{T^3}{T}\, dT$$

$$= V \int_0^T T^2 dT \quad = \frac{V}{3} T^3$$

$$\Rightarrow \quad S(T) \sim V T^3$$

$$\Rightarrow \quad S(T_1) \sim V_1 T_1^3$$

Now if $V_2 = 2V_1$, then $T_2$ will be

$$\because \quad S(T_2) - S(T_1) = 0 \sim (T_2^3 V_2 - T_1^3 V_1)$$

$$\Rightarrow \quad T_2^3 V_2 = T_1^3 V_1$$

$$T_2^3 = T_1^3 \frac{V_1}{2V_1}$$

$$\Rightarrow \quad T_2^3 = \frac{T_1^3}{2}$$

$$\Rightarrow \quad T_2 = \frac{T_1}{2^{1/3}} = \frac{3}{2^{1/3}}$$

$$\boxed{T_2 = 2.38k}$$

(b) For black body radiation, the total partition function is

$$Z = Z(k_1)\cdot Z(k_2)\cdot Z(k_2)---$$

where the partition function for the mode of wave vector $k$ is

$$Z(k) = \frac{1}{\left(1 - e^{-\hbar\omega_k\beta}\right)}$$

$$\Rightarrow \quad \langle E\rangle = -\frac{\partial}{\partial\beta}\ln(Z)$$

where, $\ln(Z) = \sum_k \ln Z(k)$

$$= \sum_k \ln\frac{1}{\left(1 - e^{-\hbar\omega_k\beta}\right)}$$

$$= \frac{-V}{(2\pi)^3}\int d^3k \; \ln\left(1 - e^{-\hbar\omega_k\beta}\right)$$

$$= \frac{-V}{(2\pi)^3}4\pi\int k^2 dk . \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

$$= \frac{-2V\,4\pi}{(2\pi)^3}\frac{1}{\hbar^3}\int_0^\infty p^2 dP \; \ln\left(1 - e^{-\beta cP}\right)$$

$$= \frac{-8\pi V}{h^3}\int_0^\infty P^2 dP \;\; \ln\left(1 - e^{-\beta cP}\right)$$

$x = \beta c\beta$

$$= -\frac{8\pi V}{h^3 (c\beta)^3}\int_0^\infty x^2 \ln(1 - e^{-x})\, dx$$

$$= +\frac{8\pi V}{(h\beta c)^3}\,\frac{1}{3}\int_0^\infty \frac{x^3}{e^x-1}\,dx$$

$$\ln(Z) = \frac{8\pi V}{(h\beta c)^3}\,\frac{\pi^4}{3(15)}$$

$$\Rightarrow \frac{\langle E\rangle}{V} = \frac{-8\pi^5}{3(15)(hc)^3}\,\frac{\partial}{\partial\beta}\,\beta^{-3}$$

$$= \frac{8\pi^5}{15(hc)^3}\,(k_B T)^4 = 10^{-14}\ J m^{-3}$$

$\because T = 2.3k.$

## Q#6

The total radiant energy flux at the Earth from Sun; is $\phi_E = 0.14\times 10^7$ erg cm$^{-2}$s$^{-1}$.

$\Rightarrow$ The flux that illuminates the Earth is propotional to the solid angle $\Delta\Omega$, subtended by the Sun's surface at the Earth:

$$\Delta\Omega = \frac{\pi R_s^2}{d^2}$$

$\Rightarrow$ The radiant energy flux at Earth is therefore;

$$\phi_E = \Delta\Omega\,\phi_s$$

$$\phi_E = \left(\frac{\pi R_s^2}{d^2}\right)\sigma_B\,T_s^4$$

$$\Rightarrow \quad \frac{0.14 \times 10^{7} \ \mathrm{erg\,cm^{-2}\,s^{-1}}}{(3.14)\ \sigma_B} \left(\frac{d}{R_s}\right)^2 = T_s^{\,4}$$

$$T_s^{\,4} = \left(\frac{1.5 \times 10^{13}}{7 \times 10^{10}}\right)^2 \cdot \frac{0.14 \times 10^{7} \ \mathrm{erg\,cm^{-2}\,s^{-1}}}{(3.14)\left(5.67 \times 10^{-5} \ \mathrm{erg\,cm^{-2}\,s^{-1}\,K^{-4}}\right)}$$

$$T_s \approx 5000\mathrm{k}$$